International Journal of Applied Operational Research
Vol. 2, No. 3, pp. 21-29, Autumn 2012

Journal homepage: www.ijorlu.ir

Optimality and Duality for an Efficient Solution of Multiobjective Nonlinear Fractional Programming Problem Involving Semilocally Convex Functions

P. Bhatnagar*

Received: 23 April 2012 ; **Accepted:** 25 August 2012

Abstract

In this paper, the problem under consideration is multiobjective non-linear fractional programming problem involving semilocally convex and related functions. We have discussed the interrelation between the solution sets involving properly efficient solutions of multiobjective fractional programming and corresponding scalar fractional programming problem. Necessary and sufficient optimality conditions are obtained for efficient and properly efficient solutions. Some duality results are established for multiobjective Schaible type dual.

Keywords: Nonlinear Programming, Multi-Objective Fractional Programming, Semilocally Convex Functions, Pseudoconvex Functions.

1. Introduction

Kaul and Kaur [1] obtained necessary optimality conditions for a non-linear programming problem by taking the objective and constraint functions to be semilocally convex and their right differentials at a point to be lower semi-continuous. Suneja and Gupta [2] established the necessary optimality conditions without assuming the semilocal convexity of the objective and constraint functions but their right differentials at the optimal point to be convex.

Suneja and Gupta [3] established necessary optimality conditions for an efficient solution of a multiobjective non-linear programming problem by taking the right differentials of the objective functions and constraint functions at the efficient point to be convex.
Consider the following Multi-objective fractional programming problem:

$$\text{(MFP)} \qquad \text{Minimize}\left[\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\dots,\frac{f_k(x)}{g_k(x)}\right],$$

$$\text{Subject to} \qquad h_j(x)\le 0,\ j=1,2,\dots,m\ ,\ x\in S\,,$$

where $S\subseteq R^n$ is a locally star shaped set and $f_i, g_i : S\to R,\ i=1,2,\dots,k$ and $h_j : S\to R,\ j=1,2,\dots,m$, are semidifferentiable functions, $f_i(x)\ge 0,\ g_i(x)>0,\ i=1,2,\dots,k$ for all $x\in S$.

* **Corresponding Author.** (✉)
E-mail: bhatnagarparas@gmail.com (P. Bhatnagar)

P. Bhatnagar
Department of Mathematics, Ideal Institute of Technology, Ghaziabad, UP, India.

Let $X^0 = \{x \in S \,|\, h_j(x) \le 0, j = 1,2,...,m\}$ and $t_i = \dfrac{f_i(x)}{g_i(x)}, i = 1,2,...,k$.

Using the parametric approach of Dinkelbach [4] and Jagannathan [5] for the scalar fractional programming and following Bector and Chandra [1] and Kaul and Lyall [6], we consider the following parametric multiobjective optimization problem (MFP_t) for $t \in R_+^k$.

(MFP_t) Minimize $\left[(f_1 - t_1 g_1)(x), (f_2 - t_2 g_2)(x), ..., (f_k - t_k g_k)(x)\right]$,
Subject to $h_j(x) \le 0,\ j = 1,2,...,m\ ,\ x \in S$.

We now have the following lemma connecting the properly efficient solutions of (MFP) and (MFP_t) .

Lemma 1. Let x^* be a properly efficient solution of (MFP), then there exists $t^* \in R_+^k$ such that x^* is properly efficient solution of (MFP_{t^*}) .Conversely if x^* is a properly efficient solution of (MFP_{t^*}) where $t_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}, i = 1,2,...,k$ then x^* is a is properly efficient solution of (MFP) .

2. Optimality conditions

Definition 1. The problem (MFP_{t^*}) is said to satisfy generalized Slater type constraint qualification at $x^* \in S$ if the functions $f_i - t_i^* g_i, i = 1,2,...,k$ and $h_j, j \in I$ are semilocally pseudoconvex at x^* and for each $r = 1,2,...,k$ there exists $x_r' \in S$ such that

$(f_i - t_i^* g_i)(x_r') < 0, i = 1,2,...,k\ ,\ i \ne r$
and $h_I(x_r') < 0$, where $I = I(x^*) = \{i \,|\, h_i(x^*) = 0\}$
and $J = J(x^*) = \{j \,|\, h_j(x^*) < 0\}$.

Now we obtain necessary optimality conditions corresponding to (MFP_{t^*}) .

Theorem 1. Let x^* be an efficient solution of (MFP) and S be convex. Let h_j be continuous at x^* for $j \in I$ and $d(f_i - t_i^* g_i)^+(x^*, x - x^*), i = 1,2,...,k$, $(dh_j)^+(x^*, x - x^*)$ be convex functions of x. If (MFP_{t^*}) satisfies generalized Slater type constraint qualification at x^* for $t_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}$, $i = 1,2,...,k$ then there exists $\xi_i^* > 0, i = 1,2,...,k$ and $\eta_j^* \ge 0,\ j \in I$ such that

$$\sum_{i=1}^{k} \xi_i^* \left[(df_i)^+(x^*, x - x^*) - t_i^*(dg_i)^+(x^*, x - x^*)\right] + \sum_{j \in I(x^*)} \eta_j^* \left[(dh_j)^+(x^*, x - x^*)\right] \ge 0, \quad (1)$$

for all $x \in S$.

Proof. We claim that the system

$$d(f_i - t_i^* g_i)^+(x^*, x - x^*) < 0, i = 1,2,...,k \ , \tag{2}$$

$$d(h_j)^+(x^*, x - x^*) < 0, \quad j \in I(x^*) \tag{3}$$

has no solution $x \in S$.
If possible let x^* be the solution of the system.
By the relation (2) we have

$$\lim_{\lambda \to 0^+} \frac{(f_i - t_i^* g_i)(x^* + \lambda(x - x^*)) - (f_i - t_i^* g_i)(x^*)}{\lambda} < 0 \ , \tag{4}$$

which implies that there exists $\delta_1 > 0$ such that

$$(f_i - t_i^* g_i)(x^* + \lambda(x - x^*)) - (f_i - t_i^* g_i)(x^*) < 0 \text{ for } \lambda \in (0, \delta_1), \tag{5}$$

by (5) , $(f_i - t_i^* g_i)(x^* + \lambda(x - x^*)) < (f_i - t_i^* g_i)(x^*)$, $i = 1,2,...,k$.

Similarly by the relation (3), there exists $\delta_2 > 0$, such that

$$h_j(x^* + \lambda(x - x^*)) - h_j(x^*) < 0, \text{ for } \lambda \in (0, \delta_2),$$

for $j \in I(x^*)$, $h_j(x^*) < 0$ and h_j is continuous at x^*,therefore there exists $\delta'_j > 0$, such that

$$h_j(x^* + \lambda(x - x^*)) < 0, \text{ for } \lambda \in (0, \delta'_j).$$

Let $\delta^* = \min\{\delta_1, \delta_2, \delta'_j\}$, then for $\lambda \in (0, \delta^*)$,we have

$$(f_i - t_i^* g_i)(x^* + \lambda(x - x^*)) < (f_i - t_i^* g_i)(x^*), i = 1,2,...,k \ ,$$
$$h_j(x^* + \lambda(x - x^*)) < 0, j \in I(x^*),$$
$$\text{and } (x^* + \lambda(x - x^*)) \in X \ .$$

This implies that x^* is not an efficient solution of (MFP_{t^*}).Hence, x^* is not an efficient solution of (MFP), contrary to the given hypothesis. Thus, the system (2), (3) has no solution x in S. Also, $d(f_i - t_i^* g_i)^+(x^*, x - x^*)$, $i = 1,2,......,k$, $(dh_j)^+(x^*, x - x^*)$, $j \in I$ are convex functions of x.

By Mangasarian [7] , there exists $\xi_i^* > 0$, $i = 1,2,...,k$,$\eta_j^* \geq 0$, $j \in I$, not all zero such that (1) holds.

We shall now show that $\xi_i^* > 0$, for each $i = 1,2,...,k$. If possible let $\xi_r^* = 0$, for some r, $1 \leq r \leq k$. By generalized Slater type constraint qualification, there exists $x'_r \in S$, such that

$(f_i - t_i^* g_i)(x'_r) < (f_i - t_i^* g_i)(x^*)$, $i = 1,2,...,k$, $i \neq r$,
$h_j(x'_r) < 0$,
and $(f_i - t_i^* g_i)$, $i = 1,2,...,k$ and h_j, $j \in I$ are semilocally pseudoconvex at x^*.

Therefore it follows that

$d(f_i - t_i^* g_i)^+(x^*, x'_r - x^*) < 0$, $i = 1,2,...,k$,
and $(dh_j)^+(x^*, x'_r - x^*) < 0$, $j \in I(x^*)$.

Since at least one of the coefficients ξ_i^*, $i = 1,2,...,k$, $i \neq r$ and η_j^*, $j \in I$ is non zero, therefore we obtain

$$\sum_{i=1}^{k} \xi_i^* \left[(df_i)^+(x^*, x'_r - x^*) - t_i^*(dg_i)^+(x^*, x - x^*) \right] + \sum_{j \in I(x^*)} \eta_j^* \left[(dh_j)^+(x^*, x'_r - x^*) \right] < 0,$$

which contradicts (1) as $\xi_i^* = 0$.
Thus $\xi_i^* > 0$, for all $i = 1,2,...,k$.

Following Geoffrion [8] we consider the scalar problem corresponding to (MFP_{t^*}) in order to prove sufficient optimality conditions.

$(\text{MFP}_{t^*})^{\xi}$ Minimize $\sum_{i=1}^{k} \xi_i (f_i(x) - t_i^* g_i(x))$,
Subject to $h_j(x) \leq 0$, $j = 1,2,...,m$.

We have the following results on the lines of Geoffrion in the light of Lemma1.
Lemma 2. If x^* is an optimal solution of $(\text{MFP}_{t^*})^{\xi}$ for some $\xi \in R^k$, with strictly positive components where $t_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}$,then x^* is a properly efficient solution of (MFP).
Theorem 2. Suppose there exists a feasible x^* for (MFP) and scalars $\xi_i^* \geq 0$, $i = 1,2,...,k$, $\eta_j^* \geq 0$, $j \in I$ such that for all $x \in X^0$,where $t_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}$, $i = 1,2,...,k$. Let the following conditions be satisfied:
(i) $f_i, -g_i$, $i = 1,2,...,k$ and h_j, $j \in I$ are semilocally convex at x^*,
(ii) $\sum_{i=1}^{k} \xi_i^* (f_i - t_i^* g_i)$ is semilocally pseudoconvex.
Then x^* is a properly efficient solution of (MFP).

Proof. Let $x \in X^0$

Then $\sum_{j\in I(x^*)}\eta_j^* h_j(x) \le \sum_{j\in I(x^*)}\eta_j^* h_j(x^*)$. (6)

(i) Since each h_j, $j \in I$ is semi locally convex at x^*, therefore it follows that $\sum_{j\in I(x^*)}\eta_j^* h_j(x^*)$ is semilocally convex. From (6) we obtain

$$\sum_{j\in I(x^*)}\eta_j^*(dh_j)^+(x^*, x-x^*) \le 0 . \quad (7)$$

Using (1) and (7), we get

$$\sum_{i=1}^{k}\xi_i^*[(df_i)^+(x^*, x-x^*) - t_i^*(dg_i)^+(x^*, x-x^*)] \ge 0, \text{ for all } x \in X^0 . \quad (8)$$

Now , $f_i, -g_i, i = 1,2,...,k$ are semilocally convex at x^*, therefore $\sum_{i=1}^{k}\xi_i^*(f_i - t_i^* g_i)$ is also semilocally convex at x^*, and (8) gives

$$\sum_{i=1}^{k}\xi_i^*(f_i(x) - t_i^* g_i(x)) \ge \sum_{i=1}^{k}\xi_i^*(f_i(x^*) - t_i^* g_i(x^*)) \text{ for all } x \in X^0 , \quad (9)$$

which implies that x^* is an optimal solution of $(\text{MFP}_{t^*})^{\xi^*}$, where ξ^* has strictly positive components. By lemma 2, it follows that x^* is properly efficient for (MFP) .
(ii) Using the semilocal convexity of $\sum_{j\in I(x^*)}\eta_j^* h_j$ at x^* in (6) we get (7) which when combined with (1) gives (8). Now $\sum_{i=1}^{k}\xi_i^*(f_i - t_i^* g_i)$ is semilocally pseudoconvex, therefore, we get (9), which implies that x^* is properly efficient for (MFP) .
In relation to (MFP) , we now associate the following multiple objective Schaible type dual program:

(MFD) Maximize $\phi(\text{t}) = \text{t}(t_1, t_2, ..t_k)$

Subject to $\sum_{i=1}^{k}\xi_i[(df_i)^+(u, x-u) - t_i(dg_i)^+(u, x-u) + \sum_{j=1}^{m}\eta_j[(dh_j)^+(u, x-u)] \ge 0$, (10)

for all $x \in X^0$,

where $\sum_{i=1}^{k}\xi_i(f_i(u) - t_i g_i(u)) \ge 0$, (11)

$$\sum_{j=1}^{m}\eta_j h_j(u) \ge 0 , \quad (12)$$

and $u \in S, \xi_i > 0, t_i \ge 0, i = 1,2,...,k$, $u_j \ge 0$, $j = 1,2,...,m$. (13)

We now prove some duality results.

3. Weak duality

Theorem 3. Let x be feasible for (MFP) and (u,ξ,η,t) be feasible for (MFD). If the following holds:

(i) $f_i, -g_i, i=1,2,...,k$ and $h_j, j \in I$ are semilocally convex at *u*,

(ii) $\sum_{i=1}^{k}\xi_i(f_i - t_i g_i)$ is semilocally pseudoconvex and $\sum_{j=1}^{m}\eta_j h_j$ is semilocally convex at *u*,

Then $f(x)$ *is not* $\leq t$

Proof. If possible let $f(x) \leq t$.

This gives $\frac{f_i(x)}{g_i(x)} \leq t_i, i=1,2,...,k \ , i \neq r$

and $\frac{f_r(x)}{g_r(x)} \leq t_r$, for some *r*.

Multiplying these inequalities by $\xi_i > 0, i=1,2,...,k$ and adding we get

$$\sum_{i=1}^{k}\xi_i(f_i(x) - t_i g_i(x)) < 0 . \tag{14}$$

Now x is feasible for (MFP) and (u,ξ,η,t) is feasible for (MFD) therefore

$$\sum_{j=1}^{m}\eta_j h_j(x) \leq \sum_{j=1}^{m}\eta_j h_j(u) .$$

Since $h_j, j=1,2,...,m$ are semilocally convex and consequently $\sum_{j=1}^{m}\eta_j h_j$ is semilocally convex at u, therefore

$$\sum_{j=1}^{m}\eta_j (dh_j)^+(u, x-u) \leq 0 .$$

Using (10), we obtain

$$\sum_{i=1}^{k}\xi_i[(df_i)^+(u,x-u) - t_i(dg_i)^+(u,x-u)] \geq 0 .$$

Now, $f_i, -g_i, i=1,2,...,k$ are semilocally convex therefore $\sum_{i=1}^{k}\xi_i(f_i - t_i g_i)$ is semilocally convex so we get

$$\sum_{i=1}^{k}\xi_i(f_i(x) - t_i g_i(x)) \geq \sum_{i=1}^{k}\xi_i(f_i(u) - t_i g_i(u)) \geq 0 .$$

This contradicts (14).
Hence $f(x)$ *is not* $\le t$.

4. Strong Duality

Theorem 4. Let x^* be feasible for (MFP) and (u^*,ξ^*,η^*,t^*) be feasible for (MFD) such that $\frac{f_i(x^*)}{g_i(x^*)}=t_i^*, i=1,2,...,k$. Let $f_i,-g_i, i=1,2,...,k$ be strictly semilocally convex and $h_j, j=1,2,...,m$ be semilocally convex at u^*, then $x^*=u^*$.

Proof. Let $x^*\ne u^*$, now x^* is feasible for (MFP) and (u^*,ξ^*,η^*,t^*) is feasible for (MFD), therefore

$$\sum_{j=1}^{m}\eta_j^* h_j(x^*)\le\sum_{j=1}^{m}\eta_j^* h_j(u^*).$$

Since $h_j, j=1,2,...,m$ are semilocally convex and consequently $\sum_{j=1}^{m}\eta_j^* h_j$ is semilocally convex at u^*, therefore

$$\sum_{j=1}^{m}\eta_j^*(dh_j)^+(u^*,x^*-u^*)\le 0.$$

Using the above inequality in the relation (10) for dual feasibility of (u^*,ξ^*,η^*,t^*) and for $x^*\in X^0$, we get

$$\sum_{i=1}^{k}\xi_i^*[(df_i)^+(u^*,x^*-u^*)-t_i^*(dg_i)^+(u^*,x^*-u^*)]\ge 0.$$

By the strict semilocal convexity of f_i and $-g_i, i=1,2,...,k$ at u^*, we obtain

$$\sum_{i=1}^{k}\xi_i^*(f_i-t_i^* g_i)(x^*)>\sum_{i=1}^{k}\xi_i^*(f_i-t_i^* g_i)(u^*),$$

$\frac{f_i(x^*)}{g_i(x^*)}=t_i^*$ gives that $\sum_{i=1}^{k}\xi_i^*(f_i(x^*)-t_i^* g_i(x^*))=0$,

therefore we get $\sum_{i=1}^{k}\xi_i^*(f_i-t_i^* g_i)(u^*)<0$,

which contradicts the fact that (u^*,ξ^*,η^*,t^*) is dual feasible.
Hence $x^*=u^*$.

Theorem 5. Let (u^*,ξ^*,η^*,t^*) be feasible for (MFD), $f_i,-g_i$, $i=1,2,...,k$ and h_j, $j=1,2,....,m$ are semilocally convex at u^*. Suppose there exists a feasible x^* for (MFP) such that $\frac{f_i(x^*)}{g_i(x^*)}=t_i^*, i=1,2,...,k$. (15)

Then x^* is properly efficient solution for (MFP). Also, if for each feasible (u,ξ,η,t) of (MFD), $f_i, -g_i, i=1,2,...,k$ and $h_j, j=1,2,...,m$ are semilocally convex at u then (u^*,ξ^*,η^*,t^*) is properly efficient solution for (MFD).

Proof. Suppose that x^* is not an efficient solution of (MFP), then there exists a feasible x for (MFP) and an index r such that

$$\frac{f_i(x)}{g_i(x)} \le \frac{f_i(x^*)}{g_i(x^*)},\ i=1,2,...,k\ ,\ i \ne r$$

and $\dfrac{f_r(x)}{g_r(x)} \le \dfrac{f_r(x^*)}{g_r(x^*)}$

Using (15), we get $\dfrac{f(x)}{g(x)} \le t^*$, which contradicts the weak duality. Therefore x^* must be an efficient solution of (MFP).

If possible let x^* be not properly efficient solution for (MFP), then by lemma 2.2.1, it follows that x^* is not a properly efficient solution of (MFP_{t^*}).

Therefore for every $M > 0$, there exists a feasible x for (MFP_{t^*}) and hence (MFP) and an index i such that

$$(f_i(x) - t_i^* g_i(x)) < (f_i(x^*) - t_i^* g_i(x^*))$$

and $\dfrac{(f_i(x^*) - t_i^* g_i(x^*)) - (f_i(x) - t_i^* g_i(x))}{(f_j(x^*) - t_j^* g_j(x^*)) - (f_j(x) - t_j^* g_j(x))} > M$.

For all j satisfying $(f_j(x) - t_j^* g_j(x)) > (f_j(x^*) - t_j^* g_j(x^*))$.

That is $t_i^* g_i(x) - f_i(x) > M(f_j(x) - t_j^* g_j(x))$ for all j satisfying

$f_j(x) - t_j^* g_j(x) > 0$.

Thus $t_i^* g_i(x) - f_i(x)$ can be made large and hence for $\xi_i^* > 0$, we get the inequality

$$\sum_{i=1}^{k} \xi_i^* (t_i^* g_i(x) - f_i(x)) > 0. \tag{16}$$

Now proceeding on similar lines in Theorem 3, we get a contradiction to (16). Therefore x^* must be properly efficient solution for (MFP). Also we can prove that (u^*,ξ^*,η^*,t^*) is properly efficient solution for (MFD).

The above result can also be proved under the assumption that for each feasible (u,ξ,η,t), $\sum_{i=1}^{k} \xi_i (f_i(x) - t_i g_i(x))$ is semilocally pseudoconvex and $\sum_{j=1}^{m} \eta_j^* h_j$ is semilocally quasiconvex.

5 Conclusion

In this paper we have obtained some results for a properly efficient solution of a multiobjective non-linear fractional programming problem involving semilocally convex and related functions by assuming generalized Slater type constraint qualification.

References

1. Bector, C. R., Chandra, S., (1987). Multiobjective fractional programming: A parametric approach. Research report faculty of management, University of Monitoba.
2. Suneja, S. K., Gupta, S., (1993). Duality in non-linear programming involving semilocally convex and related functions. Journal of Optimization, 28, 17-29.
3. Suneja, S. K., Gupta, S., (1998). Duality in multi-objective non-linear programming involving semilocally convex and related functions. European Journal of Operational Research, 107.
4. Dinkelbech, W., (1967). On non-linear fractional programming. Management Science, 13, 492-498.
5. Jagannathan, R., (1973). Duality for non-linear fractional programming. Zeitschrift- Für , Operations Research, 17, 1-3.
6. Kaul, R. N., Lyall, V., (1989). A note on non-linear fractional vector maximization. Operational Research, 26,108-121.
7. Mangasarian, O. L., (1969). Non-linear fractional Programming. Journal of Operational Research, Society of Japan, 12, 1-10.
8. Geoffrion, A. M., (1968). Proper efficiency and the theory of vector maximization. Journal of Mathematical Analysis and Applications, 22,618-630.
9. Bector, C. R., Chandra, S., Bector, M. K., (1988). Sufficient optimality conditions for a quasiconvex programming problem. Journal of optimization theory and applications, 59, 202-221.
10. Bector, C. R., Chandra, S., Bector M. K., (1989). Generalized fractional programming duality: A parametric approach. Journal of Journal of optimization theory and applications, 60, 243-260.
11. Kaul, R. N., Kaur, S., (1984). Generalized convex functions properties, optimality and duality. Technical Report, SOL, 84.
12. Kaul, R. N., Lyall, V., Kaur, S., (1988). Semilocally pseudolinearity and efficiency. European Journal of Operational Research, 36, 402-409.
13. Mangasarian, O. L., (1969). Non-linear Programming, Mc-Graw-Hill, New York.
14. Schaible, S., (1976). Fractional Programming I. Duality Management Science, 22, 858-867.