International Journal of Applied Operational Research
Vol. 5, No. 2, pp. 79-93, Spring 2015

Journal homepage: ijorlu.liau.ac.ir

Bias and Unbias Comparison between Economic Statistical Design $T^2 - VSI$ and $T^2 - FRS$ Control Charts

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Received: 19 August 2014; **Accepted:** 18 February 2015

Abstract The Hotelling's T^2 control chart, is the most widely used multivariate procedure for monitoring two or more related quality characteristics, but it's power lacks the desired performance in detecting small to moderate shifts. Recently, the variable sampling intervals (VSI) control scheme in which the length of successive sampling intervals is determined upon the preceding T^2 values has been proved to have a very good performance on detecting small to moderate shifts when it is compared to the fixed ratio sampling (FRS) T^2 control scheme. Moreover, it is shown that the VSI scheme is more economical than the FRS scheme. It is applied the cost model proposed by Lorenzen and vance (1986). This model considers the cost of producing out of control and in of control items, the cost of sampling and testing, the cost of false alarms and the cost of finding and repairing an assignable cause. Furthermore, it is assumed that the length of time that the process remains in control is exponentially distributed which allows applied the Markov chain approach for developing the cost model. It is applied genetic algorithm to determine the optimal values of model parameters by minimizing the cost function. This paper studies bias and unbias comparison between Economic Statistical design $T^2 - VSI$ and $T^2 - FRS$ control charts with respect to the expected cost per unit time.

Keywords: Hotelling's T^2 Control Chart, Economic Statistical Design (ESD), Fixed Ratio Sampling (FRS), Variable Sampling Intervals (VSI), Bias And Unbias, Genetic Algorithm(GM).

1 Introduction

Control charts are used to monitor processes to detect any change may affect the quality of the process. In many situations, quality can be characterized by a signal continuous random variable, which is usually assumed to follow a normal distribution. Here, the most common univariate control chart used in maintaining current control of the process is the Shewhart $\overline{X}$ chart [1].

On occasion, processes are characterized by several, usually correlated, variables indicating the need for the use of a multivariate control chart such as that due to Hotelling [2].

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Lowry and Montgomery [3] mentioned the popularity of the Hotelling's T^2 chart in industrial applications leading to the development of control chart software for its application [1].

The usual method of applying a control chart such at the T^2 chart to monitor a process is to obtain samples of fixed size n_0 at fixed sampling intervals h_0 between successive samples; this is referred to as Fixed Ratio Sampling (FRS).FRS control schemes have good performance in detecting large shifts in the process mean but their performance in situations in which it is necessary to detect small or even moderate as quickly as possible can be poor [1].

One procedure to improve the statistical performance of FRS control schemes is a Variable Sampling Interval (VSI) scheme that varies the sampling interval between successive samples as a function of prior sample results. In this procedure, the area between the control limits and the origin has been divided into two zones by a warning line w for the use of two different sampling intervals $(h_1 > h_2)$. If the current sample value falls in a particular zone, then the next sample is to be drawn from the process according to a particular sampling interval. The use of VSI control chart schemes requires the user to select five design parameters: the long and short sampling intervals h_1 and h_2, the fixed sample size n, the warning limit w and the control limit k [1].

Traditionally, the design of VSI schemes involves the selection of a convenient sample size and the control limit is then determined based upon a maximum probability of a type I error (false alarm) and/ or a type II error (failure to sound an alarm). The parameters w , h_1 and h_2 are determined such that the statistical performance or the speed with which process mean shifts are detected is minimized. This type of design is called statistical design. Cui and Reynolds [4] were the first to statistically design a $T^2 - VSI$ chart and showed that the time required by the chart to detect shifts in the process mean can be significantly reduced when compared to a FRS scheme. Reynolds and Arnold [5] derived expressions for the optimal one-sided Shewhart control chart subject to some constraints when the time between the samples is varying. Chengalur et al. [6] considered the multiparameter Shewhart chart with a variable sampling rate and proved that it is more efficient than the FRS scheme. Runger and Pignatiello [7], Runger and Montgomery [8], Reynolds and Arnold [9] and Bai and Lee [10] showed that the average time to signalfor the VSI chart is significantly smaller than for the FRS chart, for small to moderate shifts in the process means. Aparisi and Haro [11] presented a $T^2 - VSI$ chart in which they adopted the simplifying assumption that the process starts form an out of control state corresponding to the specific amount of shift in the process mean. Faraz and Moghadam [12] extended their work to statistically design the $T^2 - VSI$ control scheme when the shift in the process mean does not occur at the beginning but at some random time in the future. Thus they developed a model involving a prior distribution for the amount of time the process remains in control. Further, they assumed that the occurrence time of the shift is an exponentially distributed random variable [1].

All of the above studies indicated that the VSI scheme is quite effective in reducing the time to detect assignable causes in comparison with FRS scheme. Another concern is that of cost. Indeed, an alternative to statistical design is called Economic Design (ED) to which considerable attention has been placed. Montgomery [13] published a literature review on ED of control charts and listed fifty one references on the topic. Since then literally hundreds of articles on the subject have appeared. Economic design of control charts involves the optimal determination of charts parameters by minimizing the overall costs associated with maintaining current control of a process. But as Woodall [14] mentioned, the main drawback of the ED's is that they typically have a high Type I error probability, which can lead to unnecessary process adjustments or a loss of trust in the control system. Saniga [15] remedied this shortcoming by developing a design method he called Economic Statistical Design (ESD); Here statistical constraints are placed upon the cost model of ED. Using a large experiment he found that the ESD's have slightly higher costs than ED and its statistical properties are as good as statistically designed control charts. In addition ESD's have the counterintuitive property that one can, at times, reduce cost by tightening statistical constraints [1]. Montgomery and Klatt [16] have studied the ED of the T^2-FRS control chart. Taylor [17] noted that economic control charts using FRS schemes are non-optimal designs and consequently, Chou et al. [18] applied the cost model given in Montgomery and Klatt [16] to determine the values of the five parameters of the T^2-VSI control chart (i.e. the sample size, the long sampling interval, the short sampling interval, the warning limit, and the control limit) such that the expected total cost associated with the test procedure is minimized. Chen [19] proposed another ED T^2-VSI control scheme method based on Duncan's [20] model. He assumed that only distributed as well as some other restrictive assumptions [1].

We study in this paper bias and unbias comparison between Economic Statistical design T^2-VSI and T^2-FRS control charts. This paper is organized as follows: In section 2, T^2-VSI control scheme are reviewed. In section 3 unbias comparison between ESD T^2-VSI and T^2-FRS is discussed. In section 4, we have industrial example and final section provides some concluding remarks.

2 The T^2-VSI control scheme and markov chain approach

Consider a process in which p correlated characteristics are being measured simultaneously and is jointly. It is assumed that the joint probability distribution of the p quality characteristics is a $p-$variate normal distribution with in-control mean vector $\mu_0'=(\mu_{01},...,\mu_{0p})$ and variance-covariance matrix Σ. The T^2 control chart requires computing the sampling means for each of the p quality characteristics from a sample of size n. Then the subgroup statistic $T_i^2=n(\bar{x}_i-\mu_0)'\Sigma^{-1}(\bar{x}_i-\mu_0)$ is plotted on a control chart in sequential order. The chart signals as soon as $T_i^2\geq k$. If the sample value falls on a control chart limit k the process is considered in control, otherwise the process is said to be out of control and the corresponding subgroup(s) investigated [21]). Particular assumptions that govern the distribution of the T^2 statistic are separated into two cases: the parameters μ_0 and Σ of the underlying distribution being either known or unknown [22].

Case1: Assume the parameters μ_0 and Σ are known. In this case k is given by the upper α percentage point of a chi-square variable with p degrees of freedom.i.e., $k=\chi_\alpha^2(p)$.

Case 2: Assume the parameters μ_0 and Σ are unknown and $\bar{X}_i$ is independent of the estimators of μ_0 and Σ, where

$$\bar{\bar{X}}=\frac{1}{m}\sum_{j=1}^{m}\bar{X}_j\ ,\quad \bar{X}_j=\frac{1}{n}\sum_{k=1}^{n}X_{jk}\ ,\quad j=1,2,...,\mathrm{m}$$

$$\bar{S} = \frac{1}{m}\sum_{j=1}^{m} S_j \ , \ S_j = \frac{1}{n-1}\sum_{k=1}^{n}(X_{jk} - \bar{X}_j)(X_{jk} - \bar{X}_j)' \ , \ j = 1,2,...,m$$

In this case and in phase II or the process monitoring phase, if $n > 1, \frac{(m(n-1)-p+1)}{((m+1)(n-1)p)}T^2$, is distributed as F distribution with p and $(m(n-1)-p+1)$ degrees of freedom. Then each of the T_i^2 values is compared with

$$\frac{p(m+1)(n-1)}{m(n-1)-p+1}F_\alpha\left(p, m(n-1)-p+1\right) \quad (1)$$

Where $F_\alpha(v_1, v_2)$ is the upper α percentage point of F distribution with v_1 and v_2 degrees of freedom. Moreover, if $n = 1$, then we have [21].

$$k = \frac{p(m+1)(m-1)}{m(m-p)}F_\alpha\left(p, m(m-p)\right) \quad (2)$$

Upper control limit presented by Alt [23]. In this paper, we assume the parameters μ_0 and Σ are known.

The $T^2 - VSI$ control chart is a modification of the $T^2 - FRS$ control chart. Let h_1 and h_2 be maximum and minimum sampling intervals, k be control limit, respectively, such that $0 < h_2 < h_1$ and k while keeping the sample size fixed at n for administration consideration. The decision to switch between maximum and minimum sampling intervals depends on the position of the prior sample point on the control chart. If the prior sample point $(i-1)$ falls in the safe region, the maximum sampling interval h_1 and control limit k will be used for the current sample point; if the prior sample point $(i-1)$ falls in the warning region, the minimum sampling interval h_2 will be used for the current sample point. Finally, if the prior sample point falls in the action region, then the process is considered out of control. Here the safe, warning and action region are given by the warning limit w and the action limit k (safe region is given by $[0, w)$, warning region is given by $[w, k)$ and action region is given by $[k, \infty)$). The following function summarizes the control scheme of the T^2-VSI control chart:

$$h_i = \begin{cases} h_1 & if \quad 0 \le T_{i-1}^2 < w \\ h_2 & if \quad w \le T_{i-1}^2 < k \end{cases} \quad (3)$$

That h_i is present sampling interval between subgroup $(i-1)$ and subgroup (i) [22].

In the literature [24], the most recently used statistical measure to compare different control chart schemes efficiency is *AATS*. The *AATS* is the average time from the process mean shift until the chart produces a signal. This statistical measure determines the speed with which a control chart detects a process mean shift. The average time of the cycle *(ATS)* is the average time from the start of the production until the first signal the process shift. If the assignable cause occurs according to an exponential distribution with parameter λ then the expected time interval that the process remains in-control is $\frac{1}{\lambda}$ [25]. Therefore,

$$AATS = ATC - \frac{1}{\lambda} \quad (4)$$

The memoryless property of the exponential distribution allows the computation of the ATC using the Markov chain approach. The Markov chain approach employed here is similar to that of Faraz and Saniga[26]. Here, at each sampling stage, one of the following five transient states is reached according to the status of the process (in or out of control), length of the sampling interval (short or long) and quantity of the control limit k:

State 1: $0 \le T^2 < w$ and the process is in-control;
State 2: $w \le T^2 < k$ and the process is in-control;
State 3: $T^2 \le k$ and the process is in-control;
State 4: $0 \le T^2 < w$ and the process is out of control;
State 5: $w \le T^2 < k$ and the process is out of control;

The control chart produces a signal when $T^2 \ge k$. If the current state is 3, the signal is a false alarm and absorbing state (state 6) is reached when the true alarm occurs. Therefore, the transition probability matrix is given by

$$p = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where p_{ij} denotes the transition probability that i is the prior state and j is the current state. In what follows, $F(x,p,\eta)$ will denote the cumulative probability distribution function of a non-control chi-square distribution with p degrees of freedom and non-centrality parameter, $\eta = nd^2$, where

$$d^2 = (\mu_1 - \mu_0)'\Sigma^{-1}(\mu_1 - \mu_0) \quad (5)$$

Then, p_{ij}'s are

$$p_{11} = a_0 e^{-\lambda h_1},\ p_{12} = [b_0 - a_0]e^{-\lambda h_1}, p_{13} = [1-b_0]e^{-\lambda h_1}, p_{14} = a_1\left(1-e^{-\lambda h_1}\right)$$
$$p_{15} = [b_1 - a_1]\left(1-e^{-\lambda h_1}\right), p_{16} = [1-b_1]\left(1-e^{-\lambda h_1}\right), p_{21} = p_{31} = a_0 e^{-\lambda h_2}$$
$$p_{22} = p_{32} = [b_0 - a_0]e^{-\lambda h_2}, p_{23} = p_{33} = [1-b_0]e^{-\lambda h_2}, p_{24} = p_{34} = a_1\left(1-e^{-\lambda h_2}\right)$$
$$p_{25} = p_{35} = [b_1 - a_1]\left(1-e^{-\lambda h_2}\right), p_{26} = p_{36} = [1-b_1]\left(1-e^{-\lambda h_2}\right), p_{41} = p_{42} = p_{43} = 0$$
$$p_{44} = a_1, p_{45} = [b_1 - a_1], p_{46} = [1-b_1], p_{51} = p_{52} = p_{53} = 0 p_{54} = a_1$$

$p_{55} = [b_1 - a_1], p_{56} = [1 - b_1], p_{61} = p_{62} = p_{63} = p_{64} = p_{65} = 0, p_{66} = 1.$

Therefore,

$$b_1 = F(k, p, \eta = nd^2), b_0 = F(k, p, \eta = 0)$$
$$a_1 = F(w, p, \eta = nd^2), a_0 = F(w, p, \eta = 0).$$

In the at state, the expected number of trials is each state to reach the absorbing state can be obtained from $b'(I-Q)^{-1}$ where Q is the 5×5 matrix obtained from p on deleting the elements corresponding to the absorbing state, I is the identity matrix of order 5 and $b' = (p_1, p_2, p_3, 0, 0)$ is a vector of initial probabilities, with $\sum_{i=1}^{3} p_i = 1$. Hence,

$$ATC = b'(I-Q)^{-1} h \qquad (6)$$

Where $h' = (h_1, h_2, h_2, h_1, h_2)$ is the vector of sampling time intervals. In this paper assumed $b' = (0,1,0,0,0)$, for providing an extra protection and preventing problems that are encountered during start-up [21].

The cost Model

The Lorenzen and Vance [27] model is an extension of Duncan's [20] pioneering model for control chart design that allows the option of allowing production to continue during searches for and repairs of an assignable cause [1].

In building its model of a process controlled by a $T^2 - VSI$ control chart, it has made the usual assumptions about the process; these are:

1. The p quality characteristics follow a multivariate normal distribution with mean vector μ and covariance matrix Σ.
2. The process is characterized by an in-control state $\mu = \mu_0$.
3. Only one signal assignable cause produces "step change" in the process mean from $\mu = \mu_0$ to a known $\mu = \mu_1$. This results in a known value of the Mahalanobis distance.
4. "Drifting processes" are not a subject of this research. i.e., assignable causes that affect process variability are not addressed; hence it is assumed that the covariance matrix Σ is constant over time.
5. The assignable cause is assumed to occur according to a poisson process with intensity λ occurrences per hour. That is, assuming that the process begins in the in-control state, the time interval that the process remains in-control is an exponential random variable with mean $\frac{1}{\lambda}$.
6. The process is not self-correcting. That is, once a transition to an out of control state has occurred, the process can be returned to the in-control condition only by management intervention upon appropriate corrective actions.
7. The quality cycle starts with the in-control state and continues until the process is repaired after an out of control signal. It is assumed that quality cycle follows a renewal reward process [1].

Figure 1 illustrates a quality cycle observed by Duncan[20], which is divided into four time intervals of an in-control period, an out of control period, the time to take a sample and interpret the results and the time to find and repair an assignable cause.

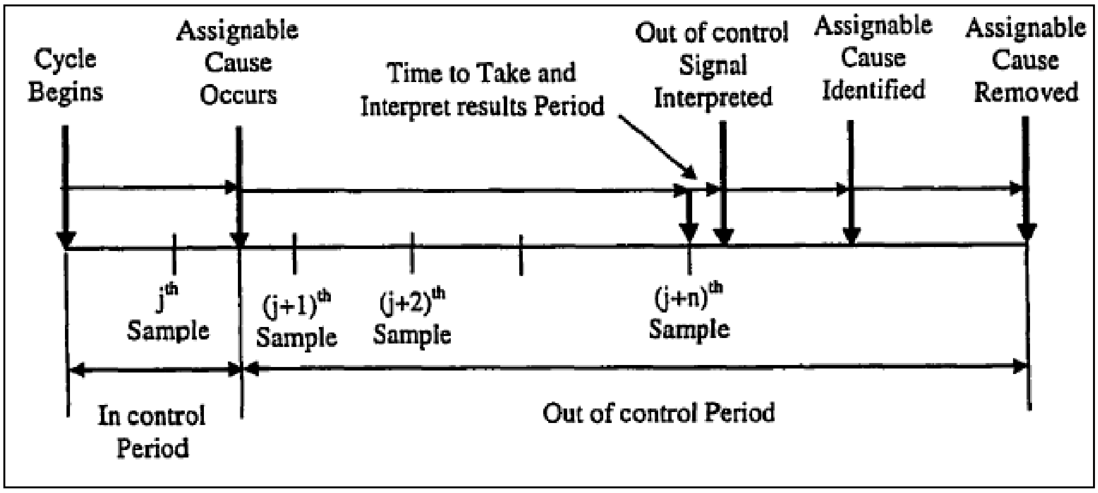

Fig.1 A Quality Cycle

The expected time interval that the process remains in-control is calculated as:

In-control period= $\frac{1}{\lambda}+\left(1-\gamma_1\right)T_0 ANF$ (7)

Where $\gamma_1=1$ if the process is not shut down during false alarms and 0 otherwise. T_0 stands for the expected time spent searching for a false alarm and ANF is the expected number of false alarms in each quality cycle and is calculated as follows:

$$ANF=b'\left(I-Q\right)^{-1}f \quad (8)$$

Where in that $f'=\left(0,0,1,0,0\right)$ [1].

The out of control period is the expected time from the process means shift until an out of control signal is triggered and is given by $AATS$. The expected time to plot and chart the sample which triggers an out of control signal is proportional to sample size and has a proportionality constant E. Therefore, the total time to take and interpret a sample is given by nE. The expected time to find the assignable cause and repair the process are given as T_1 and T_2 respectively. Therefore, the average time of a quality cycle is calculated as follows:

$$\begin{aligned} E\left(T\right) &= \frac{1}{\lambda}+\left(1-\gamma_1\right)T_0 ANF+AATS+nE+T_1+T_2 \\ &= ATC+\left(1-\gamma_1\right)T_0 ANF+nE+T_1+T_2 \end{aligned} \quad (9)$$

As mentioned by Lorenzen and Vance [27], the costs of a quality cycle can be categorized into four main components: the cost of producing non- conformities while the process is in-control, the cost of producing non-conformities while the process is out of control, the cost of evaluating alarms-both false alarms and repairing the process, and the cost of sampling [1]. If one defines C_0 and C_1 as the expected cost of producing non-conformities while the process is in control and out of control respectively, a_3' as the cost of investigating false alarms, a_3 as the cost of locating and repairing an assignable cause, a_1 and a_2 as the fixed and variable cost components of sampling and testing, respectively, then the expected cost per quality cycle, $E(C)$, is defined as:

$$E(C)=\frac{C_0}{\lambda}+C_1\left[AATS+nE+\gamma_1T_1+\gamma_2T_2\right]+a_3'ANF+a_3+\left(a_1+a_2ANI\right)+\frac{\left(a_1+a_2n\right)\left(nE+\gamma_1T_1+\gamma_2T_2\right)}{h} \quad (10)$$

where γ_2 is an indicator function for if production continues during the repair of the process the ANI and ANS stand for the expected number of inspected items and samples taken from the start of the process until the chart signals and are calculated as follows:

$$ANS=b'\left(I-Q\right)^{-1} \quad (11)$$

$$ANI=n\times ANS \quad (12)$$

It is noted that when the process goes out of control, the sampling interval h_2 is applied if the process continues. Now, based on the renewal reward process assumption, the expected cost per hour is just defined as follows:

$$E(A)=\frac{E(C)}{E(T)} \quad (13)$$

In the ESD of control charts, it is assumed that the nine process parameter (p, $(p,\lambda,T_0,T_1,T_2,\gamma_1,\gamma_2,E,d)$ and the six cost parameters $(C_0,C_1,a_1,a_2,a_3,a_3')$ are previously estimated. Then, the solution procedure finds the five chart parameter (k,w,n,h_1,h_2) which minimize (13). Among these five chart parameters, the sample size n is always a discrete variable and the other four variable are continuous where $0\le w<k$. To keep the chart practical, the minimum and maximum value of sampling intervals are considered as the possible minimum time between successive samples and maximum hours available in a work shift, respectively, i.e., $0.1\le h_2\le h_1\le 8$. Sampling intervals less than 0.1 hour may be problematic in the field. Therefore, the general optimization problem is defined as follows:

Min E(A)

s. t.

$$\begin{aligned}&0.1\le h_2\le h_1\le 8\\&0\le w\le k\\&n\in z^{+}\\&ANF\le ANF_0\\&\textit{And/or}\\&AATS\le AATS_1\end{aligned} \quad (14)$$

For offering the best protection against false alarms, the statistical constraint $ANF \leq ANF_0$ can be added to from an ESD. The optimization problem (14) has both continuous and discrete decision variables and a discontinuous and non-convex solution space. This problem can be solved with meta heuristic search techniques which at the most widely used tools in this area; examples include taboo search, simulated annealing, artificial neural network, genetic algorithms, etc [21].

The genetic algorithm approach (GA) is a method for solving both constrained and unconstrained optimization problems which is based on natural selection, the process that drives biological evolution. GA repeatedly modifies a population of individual solutions. At each step, GA selects individuals at random from the current population to be parents and uses them produce the children for the next generation. Over successive generations, the population evolves toward an optimal solution. GA can be applied to solve a variety of optimization problems that are not well suited for standard optimization algorithms, including problems in which the objective function is discontinuous, non- differentiable, stochastic, or highly nonlinear. GA has received a great deal of attention in the recent literature due to the following facts:

1. GA does not rely on analytical properties and derivative information of the function to be optimized which make it well suited to a wide class of optimization problems.
2. GA considers many points in the search space simultaneously, rather than a signal point.
3. GA works directly with strings of characters representing the parameter set, not the parameters themselves.
4. GA uses probabilistic rules, not deterministic rules to guide the search direction of finding the optimal solution. Hence, it can be applied for many kinds of optimization problems.
5. GA can lead to a global optimum by mutation and crossover techniques to refrain from trapping in a local optimum.
6. GA is able to search for many possible solutions (or chromosomes) at the same time. Hence, it can obtain the global optimal solution efficiently.

Based on these points, GA is considered as an appropriate technique for solving the optimization problem (14) and has been successfully applied to ED of control charts. For example, see Chou et al.[18], Chen[19], Faraz et al. [21] and Farazet al. [28]. The solution method applied here is the one proposed by Faraz et al. [21] which is summarized as follows:

Step 1: generate a population of size N_{pop} chromosomes to from initial generation. Each chromosome is an arbitrary solution to optimization problem (14) and usually is represented by a numerical string.

Step 2: find the expected loss per hour corresponding to each chromosome.

Step 3: scale chromosomes based on their expected loss per hour to obtain fitness values and assign each chromosome the selection probability corresponding to its fitness value. A lower expected loss per hour causes a higher fitness value and consequently the corresponding chromosome will have a higher chance for survive to next generation.

Step 4: select N_{elit} chromosomes with the best fitness values in the current generation for survive to the next generation.

Step 5: select (randomly but biased by the fitness values) two chromosomes from the mating pool of N_{pop} chromosomes. An individual can be selected more than once as a parent, in which case it contributes its genes to more than one child.

Step 6: Recombine these two chromosomes (parents) using the crossover and mutation operators to produce two new chromosomes (children). Repeat steps 5 and 6 until $N_{pop} - N_{elit}$ children are born to from the new generation.

Step 7: Repeat the steps from 2 to 6 until the termination conditions are met, i.e. when the number of generations is large enough or a satisfied fitness value is obtained [22].

3 An unbias comparison between ESD $T^2 - FRS$ and $T^2 - VSI$ scheme.

In order to performance a fair comparison between FRS and VSI schemes must to become similar the expected cost in per hour, in the state in-control for two similar schemes. When these are equal in-control time the two mentioned scheme, two schemes will be comparable, if and only if these had equal in the state in-control the cost. Because abovementioned schemes, must these had equal until are in-control process, rate sampling (sample size and sampling interval) and type-I false degree. i.e., two charts must have the same *ANF* and *ANS* and *ANI* values when the process is in-control.
The VSI design the state in-control, *ANS* in the form of below was calculating:

$$ANS_I = b'\left(I - Q\right)^{-1}\left(1,1,1,0,0\right)' \tag{15}$$

Considering the set of parameters $\left(k_0, n_0, h_0\right)$ for the FRS scheme, in the state in-cont1rol, the ANS_I value:

$$ANS_I = \frac{1}{1 - e^{-\lambda h_0}} \tag{16}$$

Considering the set of parameters (k, w, n, h_1, h_2) for the VSI scheme, the value of w is chosen so as obtain an average sampling interval h_0, while the process is in-control. Then w is determined by equating equations (15) and (16), i.e.

$$w = F^{-1}\left(\frac{\exp\left(-\lambda h_2\right) - \exp\left(-\lambda h_0\right)}{\exp\left(-\lambda h_0\right)\left(\exp\left(-\lambda h_2\right) - \exp\left(-\lambda h_1\right)\right)}, p, 0\right) \tag{17}$$

In order to have the same *ANI* values, *ANI* degree in-control have the same two schemes. For FRS scheme in-control has:

$$ANI = \frac{n_0}{1 - e^{-\lambda h_0}} = n_0 \times \frac{1}{1 - e^{-\lambda h_0}} \tag{18}$$

The VSI scheme in-control was calculated the following:

$$ANI = b'\left(I - Q\right)^{-1}\left(n, n, n, n, n\right) \tag{19}$$

Then n is determined by equating equations (18) and (19) which results is $n = n_0$.

In order to have the same *ANF* values, *ANF* degree in-control have the same two schemes, for FRS scheme in-control:

$$ANF = \left(1 - F\left(k_0, p, 0\right)\right)\frac{e^{-\lambda h_0}}{1 - e^{-\lambda h_0}} = \alpha \times \frac{e^{-\lambda h_0}}{1 - e^{-\lambda h_0}} \tag{20}$$

The VSI scheme in-control, ANF value was calculated the following:

$$ANF = b'(I-Q)^{-1}(0,0,1,0,0) \tag{21}$$

Then k is determined by equating equations (20) and (21) which results is $k=k_0$.
Therefore, the proposed procedure is as follows:
For a given process and cost parameters, the ESD of the T^2-FRS control chart is determined by optimal three chart parameters (k_0,n_0,h_0) which minimize (13). Then the two parameters k and n are set to k_0 and n_0 respectively. Hence, the goal of the ED of the T^2-VSI control chart is to find the two chart parameter h_1 and h_2 which minimize (13). The parameter w is determined by equation (17). This procedure ensures that the comparison of the two FRS and VSI scheme is meaningful and unbiased because the two procedure have the same statistical and economic performance measures while process is in-control.

4 An industrial example

In this section the proposed approach to the ESD of the T^2-VSI control chart is illustrated through on industrial example concerning the GM casting operation as presented by Lorenzen and Vance [27]. The estimated parameters are given in table1. It´s solved the optimization problem [14] with the constraint $ANF \le 0.5$ to obtain the ESD of the T^2-VSI control chart scheme and the optimal unbias design parameters are given in table 3 for different values of mean shifts d. It also solved the problem for the corresponding optimal FRS scheme that it´s estimated parameters given in table 2. The two schemes have the same in-control *ANF*, *ANI* and ANS values to guarantee a meaningful comparison; this property was discussed earlier. In table 4, unbias comparison between ESD T^2-VSI and T^2-FRS is presented.

Table 1 Estimated parameter from general motors by Lorenzen and Vance

$p=3$	$\lambda=0.05$	$\gamma_1=1$	$\gamma_2=1$	$E=0.0833$
$T_0=0.0833$	$T_0=0.0833$	$T_2=0.75$	$C_0=114.24$	$C_1=949.2$
$a_1=5$	$a_2=4.22$	$a_3=977.4$	$a_3'=977.4$	$d=1.5$

Table 2 The optimal parameters of ESD T^2-FRS scheme for different values of d

d	k_0	n_0	h_0	*ANF*	*AATS*	*E(A)*
0.25	6.87	37	2.84	0.5	10.85	546.14
0.5	8.62	18	1.34	0.5	4.47	412.81
0.75	9.31	11	0.99	0.5	2.78	349.54
1	10.2	7	0.76	0.44	2.16	314.13
1.25	11.07	5	0.63	0.35	1.81	291.21
1.5	12.23	5	0.66	0.2	1.5	273.6
1.75	12.86	4	0.6	0.16	1.32	261.08

d	k_0	n_0	h_0	*ANF*	*AATS*	*E(A)*
2	13.18	3	0.52	0.16	1.21	251.85
2.25	14.07	3	0.54	0.1	1.09	244.1
2.5	14.99	3	0.55	0.07	1	238.91
2.75	14.54	2	0.46	0.1	0.96	233.26
3	15.28	2	0.47	0.07	0.89	228.99

Table 3 The optimal parameters of unbias ESD T^2-VSI scheme for different values of d

d	k_0		n	h_1	h_2	*ANF*	*AATS*	*E(A)*
0.25	6.87	1.47	37	4.14	2.38	0.5	8.56	515.1
0.5	8.62	2.46	18	2.1	0.66	0.5	2.68	375.72
0.75	9.31	2.99	11	1.42	0.41	0.5	1.44	315.93
1	10.2	3.28	7	1.08	0.24	0.44	1.01	283.24
1.25	11.07	3.51	5	0.87	0.17	0.35	0.78	262.33
1.5	12.23	4.54	5	0.8	0.21	0.2	0.6	246.56
1.75	12.86	4.87	4	0.71	0.17	0.16	0.51	235.8
2	13.18	4.8	3	0.62	0.13	0.16	0.45	228.21
2.25	14.07	5.77	3	0.61	0.16	0.1	0.4	221.92
2.5	14.99	6.77	3	0.59	0.2	0.07	0.36	217.92
2.75	14.54	5.79	2	0.52	0.11	0.1	0.34	213.04
3	15.28	6.64	2	0.51	0.13	0.07	0.31	209.86

Table 4 Percentage decrease cost per unit time in unbias comparison

d	$E(A)_{VSI}$	$E(A)_{FRS}$	$E(A)_{FRS} / E(A)_{VSI}$	Percentage
0.25	515.1	546.14	1.0599	5%
0.5	375.72	412.81	1.0987	9%
0.75	315.93	349.54	1.1063	10%
1	283.24	314.13	1.1091	10%
1.25	262.33	291.21	1.1101	11%
1.5	246.56	273.6	1.1096	10%
1.75	235.8	261.08	1.1072	10%
2	228.21	251.85	1.1035	10%
2.25	221.92	244.1	1.0999	10%
2.5	217.92	238.91	1.0963	9%
2.75	213.04	233.26	1.0949	9%
3	209.86	228.99	1.0911	9%

If T^2-FRS and T^2-VSI schemes without equality of costs at in-control state, are compared, this will be bias comparison, that results are in table 6. Optimal bias design parameters are given in table 5. Mean percent decrease cost per unit time is presented in table 7.

Table 5 The optimal parameters of bias design ESD of the VSI scheme for different values of d

d	k	w	h_1	h_2	n	*ANF*	*AATS*	*E(A)*
0.25	5.29	0.37	6.89	5.23	48.52	0.5	8.4	508.79
0.5	6.74	0.24	4.52	2.96	24.32	0.5	3.45	378.1

d	k	w	h_1	h_2	n	ANF	$AATS$	$E(A)$
0.75	9.22	3.18	1.97	0.48	11.18	0.38	1.89	311.4
1	10.73	3.65	1.62	0.28	7.84	0.22	1.39	276.5
1.25	11.85	4.02	1.39	0.2	5.84	0.15	1.11	262.27
1.5	12.74	4.35	1.24	0.16	4.55	0.11	0.93	246.77
1.75	13.48	4.65	1.12	0.13	3.68	0.08	0.81	230.66
2	14.11	4.93	1.04	0.12	3.05	0.06	0.72	221.44
2.25	14.65	5.2	0.97	0.1	2.58	0.05	0.65	216.18
2.5	15.11	5.48	0.92	0.1	2.23	0.04	0.6	210.89
2.75	15.5	5.76	0.87	0.1	1.96	0.04	0.56	207.19
3	15.85	6.03	0.84	0.1	1.75	0.03	0.53	204.17

Table 6 Percentage decrease average cost per unit time in bias economic statistical design, T^2-FRS and T^2-VSI control charts

d	$E(A)_{VSI}$	$E(A)_{FRS}$	$E(A)_{FRS} / E(A)_{VSI}$	Percentage
0.25	508.79	546.14	1.0734	%7
0.5	378.1	412.81	1.0918	%9
0.75	311.4	349.54	1.1224	%12
1	276.5	314.13	1.1360	%13
1.25	262.27	291.21	1.1103	%11
1.5	246.77	273.6	1.1087	%10
1.75	230.66	261.08	1.1318	%13
2	221.44	251.85	1.1373	%13
2.25	216.18	244.1	1.1291	%12
2.5	210.89	238.91	1.1328	%13
2.75	207.19	233.26	1.1258	%12
3	204.17	228.99	1.1215	%12

Percentage decrease cost Geometric mean is 11 % in comparison with the ESD design, T^2-FRS with T^2-VSI.

Table7 Comparison between mean percentage decrease cost per unit time in unbias and bias designs

d	Percentage decrease mean cost in per unit time in bias design	Percentage decrease mean cost in per unit time in unbias design
0.25	7%	5%
0.5	9%	9%
0.75	12%	10%
1	13%	10%
1.25	11%	11%
1.5	10%	10%
1.75	13%	10%
2	13%	10%
2.25	12%	10%
2.5	13%	9%
2.75	12%	9%
3	12%	9%

d	Percentage decrease mean cost in per unit time in bias design	Percentage decrease mean cost in per unit time in unbias design
Geometric mean percentage decrease cost mean in per unit time	11.3239%	9.1856%

5 Concluding remarks

In this paper we have presented bias and unbias comparison between economics-statistical design T^2-VSI and T^2-FRS charts when the in control process mean vector and process covariance matrix are known. The cost model adopted in the presented study in that of Lorenzen and Vance (1986) [16] and derived by the markov chain approach. We applied the genetic algorithm to find the optimal chart parameters. The numerical comparison between the both ESD VSI and FRS schemes has shown that when we use unbias design , results show that mean percentage decrease cost per unit time in T^2-VSI scheme with respect to T^2-FRS is 0.09, while the if we use unbias design, it is 0.11, so this will lead to 2 percent error.

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