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A New Method for Ranking Extreme Efficient DMUs Based on Changing the Reference Set Using L_2 - Norm

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Abstract The purpose of this study is to utilize a new method for ranking extreme efficient decision making units (DMUs) based upon the omission of these efficient DMUs from reference set of inefficient and non-extreme efficient DMUs in data envelopment analysis (DEA) models with constant and variable returns to scale. In this method, an L_2- norm is used and it is believed that it doesn't have any existing problems of such methods. Finally, two numerical examples for illustration and comparing the proposed method with other ranking approaches are presented.

Keywords Data Envelopment Analysis (DEA), Ranking, Efficiency.

1 Introduction

Measuring the Efficiency of a decision making unit (DMU) is one of the most important objectives of data envelopment analysis (DEA). There are some methods for obtaining efficiency score of DMUs; one of them Charnes, Cooper, and Rhode's, (CCR) model [1]. Another one is a DEA ranking system based on changing the reference set proposed by Jahanshahloo et al. [2]. Several ranking methods have proposed by some authors [3-8]. Readers can be referred to Adler et al. [9] for reviewing of ranking methods. There are some methods that can be infeasible, see the Andersen and Peterson's, (AP) model [3], Mehrabian, Alirezaee, and Jahanshahloo (MAJ) [5]. The proposed approach doesn't have any problems of Andersen and Peterson's, [3] and Mehrabian's, et al. [5] models.

The structure of this paper is as follows. Section 2 describes the background of DEA. Section 3 describes the proposed method. In section 4, we extend our approach to the variable returns to scale environment. Two numerical examples are presented in section 5. Finally, in section 6 the conclusion and some remarks will be presented.

2 The background of DEA

Suppose we have n DMUs $\{DMU_j; j=1, 2, \dots, n\}$ which produce s outputs y_{rj} (r=1, 2, …, s) by utilizing m inputs x_{ij} (r=1, 2, …, m). The CCR model is the most basic DEA model that

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was proposed by Charnes et al. [1]. This model measures the efficiency of an observed DMU by the ratio output per input, i.e., how well a DMU can convert its inputs into its outputs. When we face multiple inputs and outputs for the observed DMU_p, we want to form a unique virtual output and a unique virtual input by the yet unknown weights v_i and u_r. We can obtain the weights that maximize the ratio output per input by linear programming model as follows:

$$\begin{aligned} Max \quad & \sum_{r=1}^{s} u_r y_{rp} \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\dots,n, \\ & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & v_i \ge \varepsilon, \quad i = 1,\dots,m, \\ & u_r \ge \varepsilon, \quad r = 1,\dots,s, \end{aligned} \tag{1}$$

where v_i and u_r are the weights of the input i and the output r, respectively. The dual form of model (1) is as follows:

$$\begin{aligned} Min \quad & \eta = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1,\dots,m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1,\dots,s, \\ & \lambda_j \ge 0, \quad j = 1,\dots,n, \\ & s_i^- \ge 0, \quad i = 1,\dots,m, \\ & s_r^+ \ge 0, \quad r = 1,\dots,s, \end{aligned} \tag{2}$$

where η is the measure of efficiency and ε is a non-Archimedean small and positive number. Therefore, the model (1) is feasible and then the objective function of the model (2) is bounded. We know that DMU_p is CCR-efficient if and only if in model (2) $\theta^* = 1$, $s^{-^*} = 0$ and $s^{+^*} = 0$, otherwise DMU_p is CCR-inefficient. The two-phase linear programming problem can be used for determining the CCR-efficient DMUs. Readers can refer to [10] to get further information about DEA solving procedures. Note that DMU_p is extreme efficient if and only if the model (2) has a unique optimal solution as follows:

$$\begin{aligned} & \lambda_j^* = 0, \quad j = 1,\dots,p-1,p+1,\dots,n, \\ & \lambda_p^* = 1, \\ & s_i^{-^*} = 0, \quad i = 1,\dots,m, \\ & s_r^{+^*} = 0, \quad r = 1,\dots,s. \end{aligned}$$

3 The proposed method

Suppose that we have used CCR or BCC models to obtain the efficiency score of observed DMUs and also assume that DMU_b is one of the observed DMUs. Now we omit DMU_b from the reference set of all the other DMUs so, the original efficient frontier will change if and only if DMU_b is Extreme efficient (E). The new efficient frontier (without DMU_b) gets closer to the inefficient DMUs and it is possible that some of these inefficient DMUs change to efficient.

Obviously, among the extreme efficient DMUs, the one that affects the efficient frontier to get further to the remaining DMUs should be ranked as the best one. In order to carry out our method, we re-evaluate all of the Inefficient and Non-extreme efficient (I,N) DMUs by the following model:

$$
\begin{aligned}
Min \;\; & \varphi_a^b = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
s.t. \;\; & \sum_{\substack{j=1 \\ j\neq b}}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ia}, \quad i = 1,\ldots,m, \\
& \sum_{\substack{j=1 \\ j\neq b}}^{n} \lambda_j y_{rj} - s_r^+ = y_{ra}, \quad r = 1,\ldots,s, \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n, \; j \neq b, \\
& s_i^- \geq 0, \quad i = 1,\ldots,m, \\
& s_r^+ \geq 0, \quad r = 1,\ldots,s,
\end{aligned}
\tag{3}
$$

where $a \in \Gamma_{I,N}$ and $b \in \Gamma_E$. Note that $\Gamma_{I,N}$ is the set of inefficient and non-extreme efficient DMUs and Γ_E is the set of extreme efficient DMUs.

Now we consider vector $1 = (1,1,\ldots,1)^t \in \mathbb{R}^{card(\Gamma_{I,N})}$ and call it *ideal vector*. After obtaining the measure of efficiency φ_a^b for each $a \in \Gamma_{I,N}$ by model (3), we define vector $X^{(b)} \in \mathbb{R}^{card(\Gamma_{I,N})}$ for each $b \in \Gamma_E$ as follows:

$$
X^{(b)} = (\varphi_a^b)^t \quad for\ each \;\; a \in \Gamma_{I,N}. \tag{4}
$$

Then consider:

$$
\omega^b = \left\| 1 - X^{(b)} \right\|_2 = \left(\sum_{a\in\Gamma_{I,N}} \left| 1 - \varphi_a^b \right|^2 \right)^{\frac{1}{2}}, \;\; for\ each \;\; b \in \Gamma_E. \tag{5}
$$

After calculating ω^b ($\forall b : b \in \Gamma_E$), we classify DMU_b s (the extreme efficient DMUs) based on comparing ω^b ($\forall b : b \in \Gamma_E$) as follows:

At first, we choose the smallest of ω^b s and then let its corresponding DMU_b as the first extreme efficient DMU. Now, among the rest of ω^b s, choose the smallest of them, and then let its corresponding DMU_b as the second extreme efficient DMU. Similarly, we can classify all of the extreme efficient DMU with this method. Obviously, the biggest of ω^b s is corresponding with the last of extreme efficient DMU.

4 Extension to the variable returns to scale case

So far, we discussed the ranking of extreme efficient DMUs under constant returns to scale assumption. Now we extend our discussion to the variable returns to scale case by adding the constraint $\sum_{\substack{j=1 \\ j\neq b}}^{n} \lambda_j = 1$ on the model (3). So, we have the following model:

$$
\begin{aligned}
Min \ \ & \varphi_a^b = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
s.t. \ \ & \sum_{\substack{j=1 \\ j\neq b}}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ia}, \quad i = 1,\ldots,m, \\
& \sum_{\substack{j=1 \\ j\neq b}}^{n} \lambda_j y_{rj} - s_r^+ = y_{ra}, \quad r = 1,\ldots,s, \\
& \sum_{\substack{j=1 \\ j\neq b}}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n, \ j \neq b, \\
& s_i^- \geq 0, \quad i = 1,\ldots,m, \\
& s_r^+ \geq 0. \quad r = 1,\ldots,s.
\end{aligned}
\tag{6}
$$

Similarly, we obtain $\varphi_a^b \ (\forall a : a \in \Gamma_{I,N})$ by solving model (6) and calculate $\omega^b \ (\forall b : b \in \Gamma_E)$, and then by using the method in previous case (the constant returns to scale case) we classify all of the extreme efficient DMUs.

5 Numerical examples

In this section, we present two examples with fictional and real data and in each case; we compare the proposed method with other ranking methods.

5.1 First example (Fictional data)

In this example, we are going to rank the data of table 1. Results are shown in Table 2. This table consists of five columns, the efficiency of original CCR model and the efficiency of model (3) without extreme efficient DMUs (DMUs a, b, c, and d), and also there are two rows for inefficient DMUs (DMUs e and f).

Table 1 DMUs' data (extracted from [8, p. 260])

DMU	Input 1	Input 2	Output 1	Output 2
a	150.000	0.200	14000.000	3500.000
b	400.000	0.700	14000.000	21000.000
c	320.000	1.200	42000.000	10500.000
d	520.000	2.000	28000.000	42000.000
e	350.000	1.200	19000.000	25000.000
f	320.000	0.700	14000.000	15000.000

Table 2 New efficiency evaluation

DMU	CCR	DMU_a	DMU_b	DMU_c	DMU_d
e	0.978	0.988	0.994	0.978	1.000
f	0.868	0.894	1.000	0.867	0.875
ω	-----	0.107	0.006	0.135	0.125

According to table 2, by omitting DMU_a from the reference set of all the other DMUs, none of the inefficient DMUs becomes efficient, but, by omitting DMU_b, the inefficient DMU_f becomes efficient, for instance. On the other hand, the extreme efficient DMU_b has more influence on other DMUs than the extreme efficient DMU_a has. The last row of Table 2 is calculated by using (4) and (5) and it is shown the value of ω for each extreme efficient DMUs.

In this paper, a new ranking method is presented for extreme efficient DMUs based upon the smallness of ω value in Table 2. In Table 3, we have compared the results of ranking with using the new method with several other methods.

The majority of ranking methods classify DMU_a as the best extreme efficient DMU, but our method classifies DMU_b and DMU_a as the first and second best extreme efficient DMUs, respectively.

Table 3 DMUs' scores for some ranking methods

Our results		Other ranking methods [8]									
		CCR		BCC		CEA		CEB		EDM	
b	0.006	a	1.000	a	1.000	a	0.764	a	1.000	a	200.000
a	0.106	b	1.000	b	1.000	b	0.700	d	1.000	b	140.625
d	0.125	c	1.000	c	1.000	d	0.700	e	0.974	c	140.000
c	0.133	d	1.000	d	1.000	e	0.696	b	0.955	d	133.077
e	0.978	e	0.978	e	1.000	c	0.643	c	0.886	e	97.750
f	0.868	f	0.868	f	0.896	f	0.608	f	0.847	f	860745

5.2 Second example (Real word data)

In this example, the data of 20 branch banks of Iran is evaluated by the proposed method. This data was previously analyzed by Amirteimoori and Kordrostami [11] and Jahanshahloo et al. [2] and is listed in Table 4. Results of using our approach are shown in Table 5. According to Table 5, all of the 7 CCR extreme efficient DMUs are classified by using this new method that DMU_{15} is as the best extreme efficient DMU.

Table 4 DMUs' data (extracted from [10, p. 689])

Branch	Inputs			Outputs			CCR efficiency
	Staff	Computer terminals	Space (m^2)	Deposits	Loans	Charge	
1	0.950	0.700	0.155	0.190	0.521	0.293	1.000
2	0.796	0.600	1.000	0.227	0.627	0.462	0.833
3	0.798	0.750	0.513	0.228	0.970	0.261	0.991
4	0.865	0.550	0.210	0.193	0.632	1.000	1.000
5	0.815	0.850	0.268	0.233	0.722	0.246	0.899
6	0.842	0.650	0.500	0.207	0.603	0.569	0.748
7	0.719	0.600	0.350	0.182	0.900	0.716	1.000
8	0.785	0.750	0.120	0.125	0.234	0.298	0.798
9	0.476	0.600	0.135	0.080	0.364	0.244	0.789
10	0.678	0.550	0.510	0.082	0.184	0.049	0.289
11	0.711	1.000	0.305	0.212	0.318	0.403	0.604
12	0.811	0.650	0.255	0.123	0.923	0.628	1.000
13	0.659	0.850	0.340	0.176	0.645	0.261	0.817
14	0.976	0.800	0.540	0.144	0.514	0.243	0.470
15	0.685	0.950	0.450	1.000	0.262	0.098	1.000
16	0.613	0.900	0.525	0.115	0.402	0.464	0.639
17	1.000	0.600	0.205	0.090	1.000	0.161	1.000
18	0.634	0.650	0.235	0.059	0.349	0.068	0.473
19	0.372	0.700	0.238	0.039	0.190	0.111	0.408
20	0.583	0.550	0.500	0.110	0.615	0.764	1.000

Table 5 New branch banks efficiency evaluation

DMU	CCR	DMU_{15}	DMU_4	DMU_7	DMU_{20}	DMU_{17}	DMU_{12}	DMU_1
2	0.833	1.000	0.833	0.909	0.833	0.833	0.833	0.833
3	0.991	1.000	0.991	1.000	0.991	0.991	0.991	0.991
5	0.899	1.000	0.899	0.899	0.899	0.929	0.913	0.899
6	0.748	0.950	0.810	0.812	0.748	0.748	1.748	0.748
8	0.798	0.916	1.000	0.798	0.798	0.798	0.798	0.811
9	0.789	0.824	0.816	0.789	0.789	0.808	0.814	0.789
10	0.289	0.444	0.289	0.301	0.289	0.289	1.289	0.289
11	0.604	1.000	0.754	0.612	0.614	0.604	0.604	0.604
13	0.817	0.939	0.817	0.865	0.817	0.817	0.817	0.817
14	0.470	0.560	0.470	0.514	0.470	0.470	0.470	0.470
16	0.639	0.749	0.639	0.648	0.709	0.639	0.639	0.639
18	0.473	0.478	0.473	0.484	0.473	0.473	0.483	0.473
19	0.408	0.408	0.408	0.442	0.408	0.408	0.408	0.408
ω	------	1.110	1.322	1.317	1.367	1.382	1.378	1.385

6 Conclusion

In this paper, a new method was presented to rank extreme efficient DMUs by utilizing L_2 -norm. In section 2, we briefly introduced the CCR and all the other models used in this work. Our proposed method was presented in section 3. In section 4, we extended our approach to the variable returns to scale case. Finally, in section 5 two numerical examples were presented that they were about comparing our method with some ranking methods and analyzing a real word banking data.

It seems that our proposed method is more robust than other ranking methods. Note that, other L_p - norms $(3 \leq p < \infty)$ can be used in this method and if we use L_1 - norm and L_∞ - norm then, we may obtain some extreme efficient DMUs with same ranking.

It is necessary to say that, we may have some extreme efficient DMUs that they are not in any reference set of all the other DMUs. In this case, we omit these extreme efficient DMUs from the set of observed DMUs.

Also, initial studies had shown that our method can be applied with BCC model. We suggest a further analysis in this work for future research.

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