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Evaluating Subunits Importance in Performance Measurement of Network Systems in Data Envelopment Analysis

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Abstract In conventional DEA models, decision making units (DMUs) are generally assumed as a black-box while the performance of decision making sub-units (DMSUs) and their importance play crucial roles in analyzing the performance of systems which have internal processes. The present paper introduces an ideal network which have efficient processes and next purposes a new approach for evaluating importance of network components (DMSUs) based on comparison with the ideal network. Eventually, overall efficiency of network system will be determined which can be decomposed to the weighted efficiency of its sub-processes. As the result of the purposed approach, we can determine the situations that the network would perform better by improving the efficiency of the important DMSUs which have a vital impact on network performance.

Keywords: Data envelopment analysis, Network systems, Importance of subunits, Overall efficiency

1 Introduction

Data envelopment analysis (DEA) is a methodology, developed by Charnes et al. [1], for assessing the relative efficiency of peer decision making units (DMUs) that convert multiple inputs into multiple outputs. One of the defects of these models is neglecting internal relation of production system. In real world, many DMUs have network structures and analyzing them with classical DEA models lead to non precision results. Färe and Grosskopf [2] developed several network models which can be used to discuss variations of the standard DEA model. These models have been used widely to evaluate performance of activities in which some outputs of special DMSUs are consumed by some other DMSUs as inputs.

Kao and Hwang [3] in, proposed a model for network systems with two-stage structure in which an overall efficiency of a two-stage system was decomposed into the product of the efficiencies of its two stages. Two-stage systems were defined as the systems in which the first stage some inputs to outputs (intermediate products) which are inputs of the second stage to produce final outputs. Chen et al. [4] expressed overall radial efficiency of a two-stage system as an additive weighted average of the radial efficiencies of the stages which make the

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system. Kao [5] discussed about efficiency decomposition of multi-stage systems which are the extended version of two-stage systems. Tone and Tsutsui [6] proposed a slacks-based model for evaluating the overall efficiency of the DMU, and provide a performance measure for the individual subunits of a DMU. Their proposed overall efficiency is displayed as a weighted average of the DMSUs efficiencies, where weights are exogenously imposed to show the importance of the DMSUs. Cook et al. [7] presented a method to evaluate the measure of an overall efficiency of a network system as a convex combination of its individual subunits measures. Although in this method the weights are not imposed exogenously, they are vary from one network system to another one, so, as the result they cannot properly be used to compare the performance of network systems. Kao and Chan [8] presented a multi-objective method to evaluate performance of network systems. In the method, the efficiency of each DMSU and overall efficiency of network system are calculated by different objective function in a model. Chen et al. [9] mentioned some problems of network DEA with regard to divisional efficiency and projection. They pointed that most of network DEA models have weakness in presenting sufficient projections. Also, they showed the multiplier and envelopment network DEA models are different with regard to presenting divisional efficiency and they pointed that the multiplier network DEA models should be used to determine the divisional efficiency of a network system based on its DMSUs. Kao [10] presented an efficiency decomposition for multi-syage systems in which exogenous inputs and outputs are consumed and produced in addition to intermediate products in each stage, respectively.

The current paper focuses on the derivation of an importance measure of each DMSU which form a nerwork system. Derivation of importance measure of each DMSU will result to have precise information about a network DMU and make decision makers (DMs) to decide about priority setting of improving DMSUs conveniently which lead to have an improved network system. In our proposed method DMUs assume to have multi DMSUs with general structure which apply any special network structure such as two-stage systems. To obtain the goal, we construct the ideal network which have efficient DMSUs by using, generalizing and combining some DEA models such as Foroughi [11] and Hadi-Vencheh and Foroughi [12] and then using this ideal network the importance of each DMSU is estimated by expanding the method of Castelli at al. [13] and finally the network efficiency will be measured by the Network SBM model of Tone and Tsutsui [6]. Therefore, according to the Network SBM model, the network efficiency can decompose into individual components. In fact, each component of an importance vector demonstrate the situation of its corresponding DMSU.

The rest of this paper is organized as follows. In the next section we construct the idea network model in three steps. Section 3 is devoted to presenting a method for measuring the significance of network components by the made ideal network and evaluating the overall efficiency of a network system which is based on the importance of DMSUs. Our approach is illustrated in section 4 and concluding remarks are given in section 5.

2 Constructing the ideal network

Throughout this paper, we assume a DMU (network) consists of h interdependent DMSUs. The level of external input and external output of $DMSU_k$ are denoted by x_k and y_k, respectively and fraction of output of $DMSU_i$ used as the input of the $DMSU_k$ is shown by f_{ik} $i=1,...,h$, $k=1,...,h$. Suppose $x_k > 0$ and $y_k > 0$ for all k. Therefore, $0 \le f_{ik} \le 1$. Also assume $f_{ii}=0$ for all i and F_k is the fraction of output of the $DMSU_k$ that not consumed internally within the DMU, i.e. $F_k = 1 - \sum_{i=1}^{h} f_{ik} \quad k = 1,...,h$. For ease of notation, we suppose each DMSU consume one external input and, probably, a fraction of the outputs received from the other DMSUs to produce a single output. The results obtained based on this hypothesis can be partially extended.

In this section, using the available network DMUs, the ideal network will be constructed in which all DMSUs are efficient. The following three steps are the ones which are presented to form the ideal network, respectively.

In this section, using the available network DMUs, the ideal network will be constructed in which all DMSUs are efficient. The following three steps are the ones which are presented to form the ideal network, respectively.

1. The firststep is devoted to finding the most efficient *DMSU*k , *k=1,...,h* among all DMUs. To obtain the purpose, we use the proposed model of Foroughi [11], which is introduced as follows:

$$
\begin{aligned}
& d^* = \max d_k \\
& s.t.\ \ u_k^j y_k^j - v_k^j x_k^j - \sum_{i=1}^{h} u_i^j f_{ik}^j y_i^j - t_k^j + d_k \le 0 && \forall j,k \\
& \quad -u_k^j y_k^j + v_k^j x_k^j + \sum_{i=1}^{h} u_i^j f_{ik}^j y_i^j + t_k^j \le 1 && \forall j,k \\
& \quad v_k^j x_k^j + \sum_{i=1}^{h} u_i^j f_{ik}^j y_i^j \le 1 && \forall j,k \\
& \quad \sum_{j=1}^{n} t_k^j = 1 && \forall k \\
& \quad t_k^j \in \{0,1\} && \forall j,k \\
& \quad u_k^j, v_k^j \ge 0 && \forall j,k
\end{aligned}
\tag{1}
$$

In this step, similar DMSUs of all DMUs are compared with each other and the most efficient *DMSU*k among all DMUs is determined. Model (1) is run h (number of DMSUs of each DMU) times in step 1.

2. Then, to identify a partial order of the DMSUs based on the dependence of each DMSU on the outputs of the other DMSUs, we apply a depth-first search on the directed acyclic graph (which its nodes are DMSUs and its arcs link activities between DMSUs). Then arrange the DMSUs in reverse order as reached by this search.
3. In this step, according to results of partial order from the previous step, for constructing the ideal network, the most efficient $DMSU_1$ with its own input and output (intermediate or external) is inserted as the first DMSU of the ideal network. Then, to make the remaining DMSUs of the ideal network, the most efficient $DMSU_2$ should be inserted with its external input. Note that the level of the input received by $DMSU_2$ from DMSU_1 in the ideal network may be different from the one of its own network, so, it is necessary to present a method in which the amount of outputs of $DMSU_2$ is determined based on its new set of inputs while retaining the same efficiency score on $DMSU_2$ that is equal to 1. Hence, we generalize the proposed approach of Hadi-Vencheh and Foroughi [12] for network systems to determine the outputs of $DMSU_2$ of the ideal network so that $DMSU_2$ remains efficient. For this work, we use an output-oriented DEA model as follows:

$$
\begin{aligned}
\rho_k^o = {} & \max \varphi \\
s.t. \quad & \sum_{j=1}^{n} x_k^j \lambda_k^j \le x_k^o && \forall k \\
& \sum_{j=1}^{n} f_{ik}^j y_i^j \lambda_k^j + t_k^j \le f_{ik}^o y_i^o && \forall i,k \\
& \sum_{j=1}^{n} y_k^j \lambda_k^j \ge \varphi y_k^o && \forall k \\
& \lambda_k^j \ge 0 && \forall j,k
\end{aligned}
\tag{2}
$$

Suppose the inputs of $DMSU_k$ belonging to DMU_o, are changed as follows:

$x_k^o + \Delta x_k^o = \alpha_k \qquad \Delta x_k^o \in \mathbb{R}$ (external input)

$f_{ik}^o y_i^o + \Delta x_{ik}^o = \alpha_{ik} \beta_{k-1} \qquad \Delta x_{ik}^o \in \mathbb{R}$ (intermediate input)

To have unchanged efficiency score for $DMUS_k$ as ρ_k^o, we need to estimate the output as follows:

$y_k^o + \Delta y_k^o = \beta_k \qquad \Delta y_k^o \in \mathbb{R}$

Suppose kth DMSU of DMU_{n+1} represents kth DMSU of DMU_o, after changing its inputs and outputs. Hence, to measure the efficiency of $DMSU_k$ belonging to DMU_{n+1}, we use the following model:

$$
\begin{aligned}
\rho_k^{o+} = {} & \max \varphi \\
s.t. \quad & \sum_{j=1}^{n} x_k^j \lambda_k^j + \alpha_k \lambda_k^{n+1} \le \alpha_k && \forall k \\
& \sum_{j=1}^{n} f_{ik}^j y_i^j \lambda_k^j + \alpha_{ik} \beta_{k-1} \lambda_k^{n+1} \le \alpha_{ik} \beta_{k-1} && \forall i,k \\
& \sum_{j=1}^{n} y_k^j \lambda_k^j + \beta_k \lambda_k^{n+1} \ge \varphi \beta_k && \forall k \\
& \lambda_k^j \ge 0 && \forall j,k
\end{aligned}
\tag{3}
$$

Definition 1. If the optimal value of problem (ρ_k^{o+}) be equal to the optimal value of problem (ρ_k^o), the efficiency of $DMSU_k$ will be unchanged.

In fact we are looking for the outputs of $DMSU_k$ which are produced by consuming α_k and $\alpha_{ik}\beta_{k-1}$ $\forall (i,k)$ while the efficiency score of $DMSU_k$ is preserved. To reach this aim, we apply the model (4):

$$
\begin{aligned}
v^o = {} & \max \beta_k \\
s.t. \quad & \sum_{j=1}^{n} x_k^j \lambda_k^j \le \alpha_k && \forall k \\
& \sum_{j=1}^{n} f_{ik}^j y_i^j \lambda_k^j \le \alpha_{ik} \beta_{k-1} && \forall i,k \\
& \sum_{j=1}^{n} y_k^j \lambda_k^j \ge \varphi \beta_k && \forall k \\
& \lambda_k^j \ge 0 && \forall j,k
\end{aligned}
\tag{4}
$$

Where φ is the optimal value of ρ_k^o, which is 1 in model (4), because in step 1 the most efficient $DMSU_k$ is determined for the next steps.

Similarly, we repeat this process for all the most efficient DMSUs to achieve their output based on their new set of inputs which may be depend on the output of other DMSUs and to complete the ideal network. By using this approach, we have an ideal network that all of its DMSUs are efficient.

2.1 Properties of the ideal network

The internal resources wasted in a network caused by the imbalance between supply and demand in internal processes. Note, the least internal resource waste occurs in the ideal network in comparison with the other networks, because corresponding to the third step of constructing the ideal network, the amount of internal input that each of the DMSU demand is equal to the supply of previous DMSUs.

In the following we introduce some properties of the achieved ideal network by some theorems.

Theorem 1. The introduced ideal network is unique.

Proof. Based on the previous points, the single most efficient DMSU exist for each stage in the first step of creating the ideal network. Suppose the ideal network is not unique and A and B are two ideal networks. According to assumption, $DMSU_1^A$ and $DMSU_1^B$ have the same inputs and the different outputs, but corresponding to model (4) the outputs of both $DMSU_1$s are equal, otherwise, one of the $DMSU_1$s is not efficient. Similarly, we repeat this process until to obtain the output of the network. Then, the both ideal networks are equal and it contradicts the assumption. □

Theorem 2. The ideal network is overall efficient.

Proof. According to theorem 1 of Tone and Tsutsui (2009), a network is overall efficient if and only if its all DMSUs be efficient. Thus the ideal network is overall efficient. □

Theorem 3. If networks (DMUs) treat as a black-box, the ideal network will be efficient.

Proof. It easily follows from the steps of the method. □

3 Evaluating the importance of each DMSU and the efficiency score of network system

In the first part of thissection, the importance of each DMSU will be achieved by the made ideal network in the best position of each DMSU. In the second part, forasmuch as the efficiency of a network DMU should only affected by the significance of its components, we use the achieved importance measures to evaluate efficiency of each network DMU. Note the network efficiency can decompose into its individual components, i.e., $\theta_T = \sum_{k=1}^{h} \theta_k w_k$, where θ_T is the overall efficiency of network system (DMU), θ_k is the efficiency score of $DMSU_k$ and w_k is the importance of $DMSU_k$

3.1 Evaluating the importance of each DMSU

In this part, we assess importance measure of $DMSU_k$ $w_k, k = 1,...,h$, by evaluating the relative efficiency of $DMSU_k$ in comparison with all the other DMSUs that make the ideal network. To do the evaluation, we will expand the model of Castelli et al. [13].

Castelli et al. [13] mentioned that DMSUs of one network system may be non-homogeneous, i.e., possibly they have not the same inputs and outputs, and they are interdependent, it means that the part of output produced by each of the DMSUs may be consumed by the other DMSUs. They presented nonlinear model (5) to evaluate the weights for which the efficiency of $DMSU_{k_o}$ is maximized in L.

$$\begin{aligned} h_{k_o} &= \max_{v,u,\alpha} \alpha u_{k_o} y_{k_o} \\ s.t.\ & v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i = 1 \\ & \alpha u_k y_k - (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) \le 0 \quad \forall k \in L \\ & P\varepsilon \le v_k, u_k \le PM \quad \forall k \in L \\ & \alpha > 0 \ , \ P > 0 \end{aligned} \tag{5}$$

Where u_k is the weight for the single output of $DMSU_k$ and v_k is the weight of the external input of $DMSU_k$. The weights v_k, u_k are bounded below by some $\varepsilon > 0$ and above by M. Here, L is the set of all DMSUs of the ideal network.

Now, we develop model (5) with using α_{k_o} instead of α to make it as a linear one.

$$\alpha_{k_o} = (v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i) \Big/ \sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) \quad \forall k \tag{6}$$

Theorem 4. The α_{k_o} satisfies the conditions of model (5).

By setting $\alpha_{k_o} = (v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i) \Big/ \sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i)$ model (5) becomes:

$$\begin{aligned} h_{k_o} &= \max_{v,u} u_{k_o} y_{k_o} \Big/ \sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) \\ s.t.\ & v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i = 1 \\ & (v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i) \Big/ \sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) u_k y_k - (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) \le 0 \quad \forall k \\ & P\varepsilon \le v_k, u_k \le PM \quad \forall k \\ & P > 0 \end{aligned} \tag{7}$$

Proof. First we show that $0 < \alpha_{k_o}$. We know $v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i = 1$ and $\sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) > 1$, because the former addition includes $v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i$ So, we have:

$$0 < \alpha_{k_o} = (v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i) \Big/ \sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) \le 1 \quad \forall k \in L$$

Therefore, $0 < \alpha_{k_o} = (v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i) \Big/ \sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) \quad \forall k = 1,...h$ Now, we show that the following constraint is held.

$$(v_{k_o} x_{k_o} + \sum_{i=1}^{h} u_i f_{ik_o} y_i) \Big/ \sum_{k=1}^{h} (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) u_k y_k - (v_k x_k + \sum_{i=1}^{h} u_i f_{ik} y_i) \le 0 \quad \forall k \in L$$

It is obvious that $(\alpha=1,u'_k,v'_k)$ is a feasible solution for model (5). By using the bounds which are obtained for α_{k_o} we have $\alpha_{k_o}u'_ky_k \le \alpha u'_ky_k = u'_ky_k \quad \forall k\in L.$

So,

$$\alpha_{k_o}u'_ky_k - (v'_kx_k + \sum_{i=1}^{h}u'_if_{ik}y_i) \le \alpha u'_ky_k - (v'_kx_k + \sum_{i=1}^{h}u'_if_{ik}y_i) \le 0 \quad \forall k\in L.$$

and this completes the proof. □

Theorem 5. Model (7) is equivalent to the following linear program.

$$\begin{aligned} h_{k_o} &= \max_{v,u} u_{k_o}y_{k_o} \\ s.t.\ & \sum_{k=1}^{h}(v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i) = 1 \\ & (v_{k_o}x_{k_o} + \sum_{i=1}^{h}u_if_{ik_o}y_i)u_ky_k - (v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i) \le 0 \quad \forall k\in L \\ & P\varepsilon \le v_k, u_k \le PM \quad \forall k\in L \\ & P>0 \end{aligned} \tag{8}$$

h_{k_o} shows the importance of $DMSU_{k_o}$

Proof. We know $\forall t>0$

$$\begin{aligned} h_{k_o} &= \max_{v,u} t u_{k_o}y_{k_o} \Big/ t\sum_{k=1}^{h}(v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i) \\ s.t.\ & tv_{k_o}x_{k_o} + t\sum_{i=1}^{h}u_if_{ik_o}y_i = t \\ & [t(v_{k_o}x_{k_o} + \sum_{i=1}^{h}u_if_{ik_o}y_i) \Big/ t\sum_{k=1}^{h}(v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i)]tu_ky_k - t(v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i) \le 0 \quad \forall k \\ & tP\varepsilon \le tv_k, tu_k \le tPM \quad \forall k \\ & P>0 \end{aligned} \tag{7'}$$

If (u^*,v^*) be the optimal solution of the model (7), then with the following changes $p'=tp$, $v'=tv_k, u'=tu_k$ it will be the optimal solution of the model (7′), and vice versa. So, there exist a t such that $t\sum_{k=1}^{h}(v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i)=1$ and we have

$$\begin{aligned} h_{k_o} &= \max_{v',u'} u'_{k_o}y_{k_o} \\ s.t.\ & \sum_{k=1}^{h}(v'_kx_k + \sum_{i=1}^{h}u'_if_{ik}y_i) = 1 \\ & tv_{k_o}x_{k_o} + t\sum_{i=1}^{h}u_if_{ik_o}y_i = t \\ & [t(v_{k_o}x_{k_o} + \sum_{i=1}^{h}u_if_{ik_o}y_i) \Big/ t\sum_{k=1}^{h}(v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i)]tu_ky_k - t(v_kx_k + \sum_{i=1}^{h}u_if_{ik}y_i) = \\ & [(v'_{k_o}x_{k_o} + \sum_{i=1}^{h}u'_if_{ik_o}y_i)]u'_ky_k - (v'_kx_k + \sum_{i=1}^{h}u'_if_{ik}y_i) \le 0 \quad \forall k\in L \\ & P'\varepsilon \le v'_k, u'_k \le P'M \quad \forall k \\ & P'>0 \end{aligned} \tag{7'}$$

So, models (7) and (8) (which is the same as model $(7')$) are equivalent. □

Definition 2. The vector obtained from this way is called the significance vector, because it can demonstrate the situation and the importance of each DMSU for the network.
In this method, the w_ks are independent of the network structure. Furthermore in sensitivity analysis of network, the w_ks recognizes the DMSUs which improve the performance of them, have more effect on the whole performance.

3.2 Evaluating the efficiency score of network system

For measuring the overall efficiency of the network and recognizing its components, imprimis we normalize the normal significance vector which is achieved in the previous part and then we adapt the Network SBM (input-oriented free link CRS) model of Tone and Tsutsui [6] to the data set which is used in our article as follows. In model (9), w_k is the normalized importance of DMSU$_k$.

$$\begin{aligned}
&\theta^{o*} = \min_{\lambda_k, s_k^{o-}} \sum_{k=1}^{h} w_k \left[1 - \frac{s_k^{o-}}{x_k^o}\right] \\
&s.t. \quad \sum_{j=1}^{n} x_k^j \lambda_k^j + s_k^{o-} = x_k^o \qquad \forall k \\
&\qquad \sum_{j=1}^{n} F_k^j y_k^j \lambda_k^j - s_k^{o+} = F_k^o y_k^o \qquad \forall i,k \\
&\qquad f_{kt} y_k \lambda_k = f_{kt} y_k \lambda_t \qquad \forall k,t \\
&\qquad \lambda_k, s_k^{o-}, s_k^{o+} \geq 0 \qquad \forall k
\end{aligned} \tag{9}$$

Where F_k is the fraction of the output of the $DMSU_k$ that is not consumed internally within the DMU and s_k^{o-} (s_k^{o+}) is the input (output) slack. θ^{o*} shows the overall efficiency of DMU$_o$.
Applying the optimal input slacks s_k^{o-*} of (9) we can evaluate the efficiency measure of each DMSU of DMU$_o$ by

$$\theta_k = 1 - \frac{s_k^{o-}}{x_k^o} \qquad k = 1,...,h \tag{10}$$

4 Illustrative example

To illustrate the results of our method, we apply a data set consisting of eight hypothetical DMUs (see Table 1) which are connected in four DMSUs.

First, we determine the ideal network which can be derived from the available DMUs, by our three steps' method.

In the first step, $DMSU_1$ of DMU C, $DMSU_2$ and $DMSU_3$ of DMU B and $DMSU_4$ of the DMU A are selected as most efficient DMSUs by using model (1).

In the next step, the partial order of the DMSUs based on the dependence of each DMSU on the output of other DMSUs is: $DMSU_1$, $DMSU_2$ or $DMSU_3$ and then $DMSU_4$.

Table 1. Sample data

DMU	$\frac{DMSU_1}{Input1}$	$\frac{DMSU_1}{Ouput1}$	$\frac{DMSU_2}{Input2}$	$\frac{DMSU_2}{Ouput2}$	$\frac{DMSU_3}{Input3}$	$\frac{DMSU_3}{Ouput3}$
A	4	5	5	7	3	5
B	7	8	2	7	1	3
C	3	5	6	3.5	4	9
D	9	10	10	2	12	3
E	10	9	8	3	6	2
F	8	7	15	5	10	1.5
G	7	6	4	2	7	2
H	12	5	6	4	5	3
DMU	$\frac{DMSU_4}{Input4}$	$\frac{DMSU_4}{Ouput4}$	f_{12}	f_{13}	f_{24}	f_{34}
A	10	9	2.5/5	1.5/5	1	1
B	12	4	1.6/8	3.2/8	1	1
C	11	5	0.5/5	3/5	1	1
D	15	1	3/10	5/10	1	1
E	18	1	2.5/9	4/9	1	1
F	16	2	2.5/7	3.5/7	1	1
G	15	0.5	1/6	4.5/6	1	1
H	12	2	1.5/5	3/5	1	1

Then, in the third step, constructing the ideal network is done by determination of new sets of input and output of DSMUs. Therefore, the ideal network is obtained as follows:

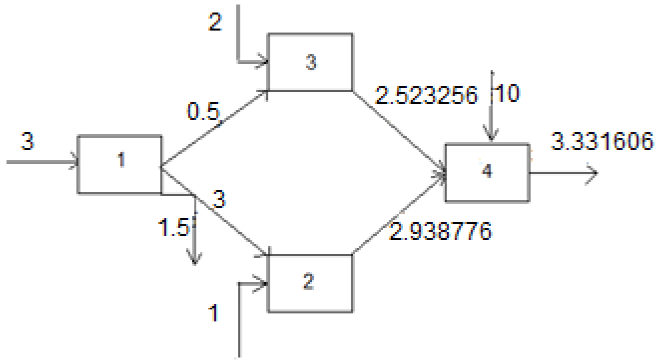

Fig. 1 Ideal network

Now, we use the ideal network which is depicted in figure 1 and model 8 to determine $w_k \ k = 1,\dots,h$. In this example the weight $\boldsymbol{u}$ and $\boldsymbol{v}$ are bounded below by 0.01 and above by 0.15. Results are summarized in table 2.

Table 2. The components of significance vector

DMSU	$DMSU_1$	$DMSU_2$	$DMSU_3$	$DMSU_4$
Significance	0.75	0.378488	0.440816	0.499741

As it can be seen in Table 2, the rank of DMSUs based on their importance is: $DMSU_1$, $DMSU_4$, $DMSU_3$ and then $DMSU_2$. The achieved importance vector will be used for all DMUs and do not vary from one DMU to another one.

The results of the network SBM model are displayed in Table 3. $\boldsymbol{w}=(w_1,w_2,w_3,w_4)=(0.362486,0.182929,0.213053,0.241532)$ is the normalized significance vector of DMSUs, which is obtained in table2 and is satisfied in $\theta_T=\sum_{k=1}^{h}\theta_k w_k$. As it is illustrated, the overall efficiency of DMU B and DMU C is one, because the efficiency scores of its components are one, and the sum of the ***w's*** components must be equal to one. Also, the result is established for all other DMUs, and as it can be seen all the DMSUs have an effect in the performance of the whole DMU, because none of the components of ***w*** is zero.

Table 3. Results of the network SBM model

DMU	Overall score	θ_1	θ_2	θ_3	θ_4
A	0.75928	0.792308	0.4	0.738669	1
B	1	1	1	1	1
C	1	1	1	1	1
D	0.284467	0.486111	0.329427	0.052083	0.152778
E	0.348725	0.544570	0.307377	0.136612	0.273224
F	0.149018	0.269097	0.05487	0.037037	0.138889
G	0.187578	0.285714	0.104167	0.089286	0.190171
H	0.111011	0.097863	0.074074	0.081007	0.185185

5 Conclusion

Traditional DEA models ignore the internal structures and the intermediate products of DMUs. So, these models are not able to evaluate the efficiency score of network systems accurately. Also they can not reflect the effect of DMSUs on performance of DMU. In this paper, the problem of generalizing DEA models in order to assess the significance of a set of non-homogeneous and interdependent DMSUs of the network is presented. To achieve this goal, we develop some presented models for systems with network structure and combine them to make the ideal network with efficient DMSUs. Then, ideal network is used to dtermination of importance of each DMSU. Finally, we compute the efficiency score of network system which is based on importance of its maker DMSUs. The importance of DMSUs identified in this paper is independent of networks' data and play crucial role in improving the network performance.

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