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Scale Efficient Targets in Systems with Two-Stage Structure under Imprecise Data Assumption

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Abstract Traditional data envelopment analysis (DEA) models evaluate two-stage decision making unit (DMU) as a black box and neglect the connectivity may exist among the stages. This paper looks inside the system by considering the intermediate activities between the stages where the first stage uses inputs to produce outputs which are the inputs to the second stage along with its own inputs. Additionally, some of the inputs and outputs values may not be completely available because of uncertainty. Data can be interval e.g. when the missing values are replaced by intervals in which the unknown values are likely to belong. We introduce models to optimize two-stage DMU with interval data. Numerical example is applied to clarify the models.

Keywords: Data Envelopment Analysis (DEA), Network DEA, Interval Data, Most Productive Scale Size (MPSS), Scale Efficient Target.

1 Introduction

In decision-making units, which use multiple inputs to produce multiple outputs, managers make decisions about how to use, integrate and process the inputs and resources. Managers tend to improve the values of inputs to obtain the most productivity. Identification of the smallest and the largest projects with the most productive scale size for a decision-making unit shows the necessary amounts of increase or decrease in input values to obtain the most productivity. The most productive scale size projects are called scale efficient targets. In multi-stage structure units, such as production and industrial units, scale efficient target must be set for each stage and for overall process as well. To do so, the connectivity and the interrelationship among the stages must be considered. This paper deals with productivity management in decision-making units with two-stage structure by using data envelopment analysis technique.

Data envelopment analysis (DEA) is a nonparametric technique based on mathematical programming to evaluate performance of homogenous multi input/output decision making units. There are many decision making units with network structure in which the outputs of one division or sub-process are the inputs to another sub-process. Banks have such network structure where labor, physical capital, and financial equity capital are inputs of the first stage to raise deposits, which are as intermediate outputs. In the second stage, banks use the deposits raised from the first stage to produce loans and security investments.

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Färe and Primont [1] applied a DEA approach for the first time to the performance evaluation of multi-plant firms modeled as DMUs with multi-stage structure. Färe and Grosskopf [2, 3] suggested models to measure efficiency scores of multi-stage DMUs in static and dynamic cases. In dynamic case, activities of DMU in one period affect the ones in the next period.

There have been many studies dealing with systems with two-stage structure. Wang et al. [4] proposed a DEA model to measure the efficiency score of two-stage structure DMUs without considering the intermediate products. Seiford and Zhu [5] extended their approach and applied modified model to assess the efficiency of the top 55 US commercial banks. Chen and Zhu [6] improved the models presented by Seiford and Zhu by considering the intermediate products to project two-stage structure DMUs on efficient frontier. Kao and Hwang [7] evaluated efficiency score of two-stage DMU as the product of efficiencies of stages. Chen et al. [8] measured the efficiency score of two-stage structure DMU as a weighted mean of efficiency scores of stages. Yang et al. [9] presented a non-linear programming to measure the efficiency of two-member supply chains, as two-stage DMUs. Paradi et al. [10] developed a two-stage DEA approach for simultaneously benchmarking the performance of operating units and a modified slacks-based measure model to aggregate the obtained efficiency scores from stage one and generate a composite performance index for each unit. Fukuyama and Mirdehghan [11] proposed slack-based network approach for identifying the efficiency status of each DMU and its divisions. Amirteimoori [12] and Liu [13] proposed DEA approaches for performance assessment of two-stage decision process in existence of imperfect outputs and Fuzzy data respectively. Wang et al. [14] utilized the network DEA approach to evaluate the efficiencies of major Chinese commercial banks. Barros and Wanke [15] presented an efficiency assessment of African airlines, using the TOPSIS – Technique for Order Preference by Similarity to the Ideal Solution. TOPSIS is a multi-criteria decision making technique, which similar to DEA, ranks a finite set of units based on the minimization of distance from an ideal point, and the maximization of distance from an anti-ideal point. In this research, TOPSIS was used first in a two-stage approach, in order to assess the relative efficiency of African airlines using the most frequent indicators adopted by the literature on airlines. Fathalikhani [16] proposed a two-stage DEA model considering simultaneously the structure of shared inputs, additional input in the second stage and part of intermediate products as the final output. Kazemi Matin et al. [17] introduced an ideal network which have efficient processes and purposed a new approach for evaluating importance of network components (DMSUs) based on comparison with the ideal network. Koushki [18] presented a dynamic DEA network approach to evaluate two-stage structure DMUs where the activity and the performance of DMU in one period effect on its efficiency in the next period. According to the results of proposed dynamic model, the inefficiencies of DMUs improve considerably.

DEA models improve input (and output) values of DMUs by radial and non-radial approaches. However, traditional DEA models evaluate multi-stage DMU as a black box and neglect the connectivity may exist among the stages. We look inside the system and introduce models to optimize two-stage DMU by considering the intermediate activities between the stages. This paper presents radial and non-radial models to measure efficiency scores of two-stage structure DMUs in the cases that internal activities are assumed fixed and non-fixed.

One of the most important concepts about these systems is identifying the most productive scale size (MPSS) pattern. A production possibility $(X_o, Y_o) \in T$ represents MPSS for its specific mix of inputs/outputs if and only if for all $(\alpha X_o, \beta Y_o) \in T$ we have $\alpha \geq \beta$ (see Banker and Thrall [19], and Banker [20]). In other words, a production possibility is not

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MPSS when either (a) all outputs can be increased in proportions that are at least as great as the corresponding proportional increases in all inputs needed to bring them about, or (b) all inputs can be decreased in proportions that are at least as great as the accompanying proportional reduction in all outputs.

Additionally, some of the inputs and outputs values of DMUs may not be completely available because of uncertainty. In imprecise data envelopment analysis (IDEA) the data can be interval e.g. when the missing values are replaced by intervals in which the unknown values are likely to belong. DEA models were initially applied by Cooper et al. [21, 22 and 23] to evaluate the performance of DMUs with interval data. Entani et al. [24], Despotis and Smirlis [25], Wang et al. [26] and Toloo et al. [27] proposed models to determine the lower and upper bounds of the efficiency scores for each DMU. Mostafaee and Saljooghi [28] presented a method to obtain the lower bound and upper bound of cost efficiency in the presence of interval data.

This paper proposes models for the first time to measure the efficiency score and to identify the most productive scale size (MPSS) pattern of two-stage structure DMUs, with interval data, where the outputs of the first stage are the inputs of the second stage along with its own inputs. Section 2.1 presents models to evaluate the performance of two stage structure DMUs. In section 2.2, the concept of MPSS in two stage structure DMU is defined. Additionally, models to determine the scale efficient targets are proposed and are applied in the case of interval data in section 2.3. Section3 contains numerical example to clarify the proposed models.

2 Network DEA

Let X_j, Y_j, Z_j and W_j be the m -dimensional input, s -dimensional final output vectors, vector associated with p -dimensional intermediate output and q -dimensional input, respectively, of DMU_j (j = 1,..,n) . X_o and Z_o are the input and the output vectors respectively of stage1 of DMU_o ;In stage2 Z_o, W_o and Y_o are the input and the output vectors respectively (Figure 1).

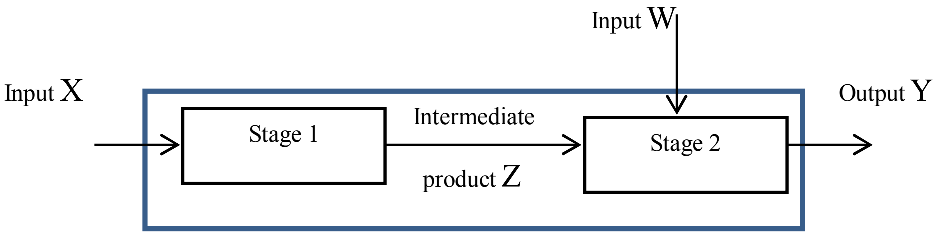

Fig.1 DMU with two-stage structure

Production possibility set (PPS) T_N is defined as follow:

$$T_N = \left\{ (X,Z,W,Y) \mid X \geq \sum_{j=1}^{n} \lambda_j^1 X_j, Z = \sum_{j=1}^{n} \lambda_j^1 Z_j, Z = \sum_{j=1}^{n} \lambda_j^2 Z_j, W \geq \sum_{j=1}^{n} \lambda_j^2 W_j, Y \leq \sum_{j=1}^{n} \lambda_j^2 Y_j, \lambda_j^k \geq 0, j = 1,...,n \ ; k = 1,2 \right\}$$

Equalities $Z = \sum_{j=1}^{n} \lambda_j^1 Z_j$ and $Z = \sum_{j=1}^{n} \lambda_j^2 Z_j$ represent the series relationship and connectivity between stages.

2.1 Performance evaluation

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In this part of section2, radial and non-radial models to measure the efficiency scores of DMUs with two stage structure are presented. Proposed models will be applied in pessimistic and the optimistic viewpoints. In the optimistic viewpoint, a DMU under evaluation is in its best situation whilst in the pessimistic viewpoint; a DMU which is under evaluation is in its worst situation. In interval efficiency assessment the final efficiency score for each DMU is characterized by an interval. The efficiency scores in the pessimistic and the optimistic viewpoints are the lower and upper bounds of this interval, respectively.
According to the definition of the PPS T_N, follow models, in radial and non-radial cases, are proposed to measure the efficiency scores of DMUs.

1) Radial models
In the following proposed models, the radial reduction of inputs and the radial increment of outputs are denoted by γ_1, γ_2 respectively.
a) Values of intermediate products are fixed. On this assumption, the efficiency score of DMU_o is measured by solving the following model:

$$\begin{aligned} Min \quad & \frac{\gamma_1}{\gamma_2} \\ s.t. \quad & \\ & (\gamma_1 X_o, Z_o, \gamma_1 W_o, \gamma_2 Y_o) \in T_N \end{aligned} \tag{1}$$

Model (1) is the follow programming:

$$\begin{aligned} Min \quad & \frac{\gamma_1}{\gamma_2} \\ s.t. \quad & \\ & \sum_{j=1}^{n} \lambda_j^1 X_j \le \gamma_1 X_o \\ & \sum_{j=1}^{n} \lambda_j^1 Z_j = Z_o \\ & \sum_{j=1}^{n} \lambda_j^2 Z_j = Z_o \\ & \sum_{j=1}^{n} \lambda_j^2 W_j \le \gamma_1 W_o \\ & \sum_{j=1}^{n} \lambda_j^2 Y_j \ge \gamma_2 Y_o \\ & \gamma_1 \le 1, \gamma_2 \ge 1, \lambda^k \ge 0 \qquad k = 1,2. \end{aligned} \tag{2}$$

$\{\lambda_o^k = 1, \lambda_j^k = 0, \gamma_1 = 1, \gamma_2 = 1\}$ for $k = 1,2,\ j = 1,..,n,\ j \ne o$ is a feasible solution of model (2).

Definition1. (X_o, Z_o, W_o, Y_o) is efficient iff $\gamma_1^* = 1, \gamma_2^* = 1$.
Theorem2. Production possibility $(\gamma_1^* X_o, Z_o, \gamma_1^* W_o, \gamma_2^* Y_o)$ is efficient.

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Proof. Let $\bar{\gamma}_1, \bar{\gamma}_2$ be the optimal values of γ_1, γ_2 obtained by solving the following model:

$$
\begin{aligned}
& Min \quad \frac{\gamma_1}{\gamma_2} \\
& s.t. \\
& \quad \sum_{j=1}^{n} \lambda_j^1 X_j \le \gamma_1(\gamma_1^* X_o) \\
& \quad \sum_{j=1}^{n} \lambda_j^1 Z_j = Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j^2 Z_j = Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j^2 W_j \le \gamma_1(\gamma_1^* W_o) \\
& \quad \sum_{j=1}^{n} \lambda_j^2 Y_j \ge \gamma_2(\gamma_2^* Y_o) \\
& \quad \gamma_1 \le 1, \gamma_2 \ge 1, \lambda^k \ge 0 \qquad\qquad k = 1,2.
\end{aligned}
\tag{3}
$$

If $\bar{\gamma}_1 < 1$, from $\bar{\gamma}_2 \ge 1$ we have $\frac{\bar{\gamma}_1 \gamma_1^*}{\bar{\gamma}_2 \gamma_2^*} < \frac{\gamma_1^*}{\gamma_2^*}$; which is a contradiction. Similar contradiction is obtained by assumption $\bar{\gamma}_2 > 1$.■

Dividing all constraints of model (2) by γ_2 results in the following linear model:

$$
\begin{aligned}
& Min \quad \omega \\
& s.t. \sum_{j=1}^{n} \lambda_j'^1 X_j \le \omega X_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^1 Z_j = \omega_1 Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^2 Z_j = \omega_1 Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^2 W_j \le \omega W_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^2 Y_j \ge Y_o \\
& \quad \omega \le \omega_1, \omega_1 \le 1 \\
& \quad \lambda'^k \ge 0 \qquad\qquad k = 1,2.
\end{aligned}
\tag{4}
$$

Where $\lambda_j'^k = \frac{1}{\gamma_2}\lambda_j^k, \omega = \frac{\gamma_1}{\gamma_2}, \omega_1 = \frac{1}{\gamma_2} \quad k = 1,2$. Therefore, according to deffinition1, (X_o, Z_o, W_o, Y_o) is efficient iff in optimality $\omega^* = 1, \omega_1^* = 1$.

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b) Values of intermediate products are not fixed. On this assumption, the efficiency score of DMU_o is measured by solving the following model:

$$\begin{aligned} &Min \quad \frac{\gamma_1}{\gamma_2} \\ &s.t. \\ &\qquad (\gamma_1 X_o, \alpha Z_o, \gamma_1 W_o, \gamma_2 Y_o) \in T_N . \end{aligned} \tag{5}$$

α represents the possible change of intermediate activities when optimizing DMU_o.
Model (5) is the follow programming:

$$\begin{aligned} &Min \quad \frac{\gamma_1}{\gamma_2} \\ &s.t. \\ &\sum_{j=1}^{n} \lambda_j^1 X_j \le \gamma_1 X_o \\ &\sum_{j=1}^{n} \lambda_j^1 Z_j = \alpha Z_o \\ &\sum_{j=1}^{n} \lambda_j^2 Z_j = \alpha Z_o \\ &\sum_{j=1}^{n} \lambda_j^2 W_j \le \gamma_1 W_o \\ &\sum_{j=1}^{n} \lambda_j^2 Y_j \ge \gamma_2 Y_o \\ &\alpha > 0, \gamma_1 \le 1, \gamma_2 \ge 1, \lambda^k \ge 0 \qquad\qquad k = 1,2. \end{aligned} \tag{6}$$

$\{\lambda_o^k = 1, \lambda_j^k = 0, \gamma_1 = 1, \gamma_2 = 1, \alpha = 1\}$ for $k = 1,2,\ j = 1,..,n,\ j \ne o$ is a feasible solution of model (6).

Definition3. (X_o, Z_o, W_o, Y_o) is efficient iff $\gamma_1^* = 1, \gamma_2^* = 1$.
Theorem4. Production possibility $(\gamma_1^* X_o, \alpha^* Z_o, \gamma_1^* W_o, \gamma_2^* Y_o)$ is efficient.
Proof. The proof is similar to the proof of theorem2.

Dividing all constraints of model (6) by γ_2 results in the following linear model:

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$$
\begin{aligned}
&Min \quad \omega \\
&s.t. \\
&\sum_{j=1}^{n} \lambda_j'^1 X_j \le \omega X_o \\
&\sum_{j=1}^{n} \lambda_j'^1 Z_j = \alpha' Z_o \\
&\sum_{j=1}^{n} \lambda_j'^2 Z_j = \alpha' Z_o \\
&\sum_{j=1}^{n} \lambda_j'^2 W_j \le \omega W_o \\
&\sum_{j=1}^{n} \lambda_j'^2 Y_j \ge Y_o \\
&\alpha' > 0, \lambda'^k \ge 0 \qquad\qquad k = 1,2.
\end{aligned} \tag{7}
$$

Where $\lambda_j'^k = \dfrac{1}{\gamma_2}\lambda_j^k, \omega = \dfrac{\gamma_1}{\gamma_2}, \alpha' = \dfrac{1}{\gamma_2}\alpha \qquad k = 1,2$. To determine the values of γ_1, γ_2, a method in the next section will be presented.

2) Non-radial model

A non-radial slacks-based measure (SBM) model to measure the efficiency score of DMU_o, by considering the connectivity between stages as equalities $\sum_{j=1}^{n} \lambda_j^1 Z_j = \sum_{j=1}^{n} \lambda_j^2 Z_j$, is proposed as follows:

$$
\begin{aligned}
&Min \quad \rho = \frac{1 - \frac{1}{m+q}\left(\sum_{i=1}^{m} \frac{s_i^{-1}}{x_{io}} + \sum_{t=1}^{q} \frac{s_t^{-2}}{w_{to}}\right)}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+}}{y_{ro}}} \\
&s.t. \\
&\sum_{j=1}^{n} \lambda_j^1 x_{ij} = x_{io} - s_i^{-1} \qquad i = 1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j^1 Z_j = \sum_{j=1}^{n} \lambda_j^2 Z_j \\
&\sum_{j=1}^{n} \lambda_j^2 w_{tj} = w_{to} - s_t^{-2} \qquad t = 1,\dots,q \\
&\sum_{j=1}^{n} \lambda_j^2 y_{rj} = y_{ro} + s_r^{+} \qquad r = 1,\dots,s \\
&\lambda^k \ge 0 \qquad k = 1,2 \\
&s_i^{-1} \ge 0, s_t^{-2} \ge 0, s_r^{+} \ge 0 \qquad i = 1,\dots,m \qquad r = 1,\dots,s \qquad t = 1,\dots,q.
\end{aligned} \tag{8}
$$

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Definition5. (X_o, Z_o, W_o, Y_o) is efficient iff in optimality we have $S^{-1^*} = 0, S^{-2^*} = 0, S^{+^*} = 0$.

Theorem6. Production possibility $(X_o - S^{-1^*}, Z_o, W_o - S^{-2^*}, Y_o + S^{+^*})$ is efficient.

Proof. We apply model (8) to evaluate the efficiency of $(X_o - S^{-1^*}, Z_o, W_o - S^{-2^*}, Y_o + S^{+^*})$ as follows:

$$Min \quad \rho = \frac{1 - \frac{1}{m+q}\left(\sum_{i=1}^{m} \frac{s_i^{-1}}{x_{io}} + \sum_{t=1}^{q} \frac{s_t^{-2}}{w_{to}}\right)}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+}}{y_{ro}}}$$

s.t.

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda_j^1 x_{ij} = x_{io} - s_i^{-1*} - s_i^{-1} && i = 1,\ldots,m \\
&\sum_{j=1}^{n} \lambda_j^1 Z_j = \sum_{j=1}^{n} \lambda_j^2 Z_j && \\
&\sum_{j=1}^{n} \lambda_j^2 w_{tj} = w_{to} - s_t^{-2*} - s_t^{-2} && t = 1,\ldots,q \\
&\sum_{j=1}^{n} \lambda_j^2 y_{rj} = y_{ro} + s_r^{+*} + s_r^{+} && r = 1,\ldots,s \\
&\lambda^k \geq 0 && k = 1,2 \\
&s_i^{-1} \geq 0, s_t^{-2} \geq 0, s_r^{+} \geq 0 && i = 1,\ldots,m \quad r = 1,\ldots,s \quad t = 1,\ldots,q.
\end{aligned} \tag{9}$$

Let $\overline{S}^{-1} = (\overline{s}_1^{-1},\ldots,\overline{s}_m^{-1}), \overline{S}^{-2} = (\overline{s}_1^{-2},\ldots,\overline{s}_q^{-2}), \overline{S}^{+} = (\overline{s}_1^{+},\ldots,\overline{s}_s^{+})$ be optimal vectors corresponding to the vectors $S^{-1} = (s_1^{-1},\ldots,s_m^{-1}), S^{-2} = (s_1^{-2},\ldots,s_q^{-2}), S^{+} = (s_1^{+},\ldots,s_s^{+})$ in model (9).

If there exists $i' \in \{1,\ldots,m\}$ satisfying $\overline{s}_{i'}^{-1} > 0$, then vectors

$(s_1^{-1^*} + \overline{s}_1^{-1},\ldots,s_{i'}^{-1^*} + \overline{s}_{i'}^{-1},\ldots,s_m^{-1^*} + \overline{s}_m^{-1}),(s_1^{-2^*} + \overline{s}_1^{-2},\ldots,s_q^{-2^*} + \overline{s}_q^{-2}),(s_1^{+^*} + \overline{s}_1^{+},\ldots,s_s^{+^*} + \overline{s}_s^{+})$

result in the objective less than the objective obtained based on the vectors $(s_1^{-1},\ldots,s_m^{-1}),(s_1^{-2},\ldots,s_q^{-2}),(s_1^{+^*},\ldots,s_s^{+^*})$. This is a contradiction because of the optimality of the vectors $S^{-1^*} = (s_1^{-1^*},\ldots,s_m^{-1^*}), S^{-2^*} = (s_1^{-2^*},\ldots,s_q^{-2^*}), S^{+^*} = (s_1^{+^*},\ldots,s_s^{+^*})$. ■

2.2 Scale efficient targets

This part of section2 includes: 1) definition of MPSS, based on the input and output vectors, in two stage structure DMUs, 2) approaches to identify MPSS and to project DMUs on MPSS, as scale efficient target, 3) applying proposed models in the case of interval data, and 4) numerical example to clarify the models.

Definition7. Production possibility $(X, Z, W, Y) \in T_{NV}$ with

$$T_{NV} = \left\{ (X,Z,W,Y) \mid X \geq \sum_{j=1}^{n} \lambda_j^1 X_j, Z = \sum_{j=1}^{n} \lambda_j^1 Z_j, Z = \sum_{j=1}^{n} \lambda_j^2 Z_j, W \geq \sum_{j=1}^{n} \lambda_j^2 W_j, Y \leq \sum_{j=1}^{n} \lambda_j^2 Y_j, 1\lambda^k = 1, \lambda^k \geq 0, k = 1,2 \right\}$$

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is MPSS if and only if for all $(\alpha_1 X, \eta Z, \alpha_1 W, \beta Y) \in T_{NV}$ we have $\alpha_1 \geq \beta$.

Suppose the values of intermediate products are not fixed. By this assumption, we consider model (6) to evaluate the efficiency scores of DMUs. Additionally, there are series relationships between the stages of each DMU. To hold this connectivity in scale efficient target of each DMU, as it will be shown, the equalities $\sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2$ should be considered. Therefore, model (6) becomes as follows:

$$
\begin{aligned}
Min \quad & \frac{\gamma_1}{\gamma_2} \\
s.t. \quad & \\
& \sum_{j=1}^{n} \lambda_j^1 X_j \leq \gamma_1 X_o \\
& \sum_{j=1}^{n} \lambda_j^1 Z_j = \alpha Z_o \\
& \sum_{j=1}^{n} \lambda_j^2 Z_j = \alpha Z_o \\
& \sum_{j=1}^{n} \lambda_j^2 W_j \leq \gamma_1 W_o \\
& \sum_{j=1}^{n} \lambda_j^2 Y_j \geq \gamma_2 Y_o \\
& \sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 \\
& \alpha > 0, \gamma_1 \leq 1, \gamma_2 \geq 1, \lambda^k \geq 0 \qquad k = 1,2.
\end{aligned}
\tag{10}
$$

Theorem8. Let γ_1^*, γ_2^* be the optimal values of γ_1, γ_2 obtained by solving model (10) to evaluate the efficiency of $(X, Z, W, Y) \in T_{NV}$.

$(X, Z, W, Y) \in T_{NV}$ is MPSS iff $\gamma_1^* = 1, \gamma_2^* = 1$.

Proof. Let $\{\lambda^{k*}\}_{k=1}^{2}$ satisfy the following constraints:

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$$
\begin{aligned}
&\sum_{j=1}^{n} \lambda_j^{1*} X_j \le \gamma_1^* X \\
&\sum_{j=1}^{n} \lambda_j^{1*} Z_j = \alpha^* Z \\
&\sum_{j=1}^{n} \lambda_j^{2*} Z_j = \alpha^* Z \\
&\sum_{j=1}^{n} \lambda_j^{2*} W_j \le \gamma_1^* W \\
&\sum_{j=1}^{n} \lambda_j^{2*} Y_j \ge \gamma_2^* Y \\
&\sum_{j=1}^{n} \lambda_j^{1*} = \sum_{j=1}^{n} \lambda_j^{2*} \\
&\alpha^* > 0, \gamma_1^* \le 1, \gamma_2^* \ge 1, \lambda^{k*} \ge 0 \qquad k = 1,2.
\end{aligned} \tag{11}
$$

Let $\sum_{j=1}^{n} \lambda_j^{k^*} = \delta \quad k = 1,2$. Dividing above constraints by δ results in the following:

$$
\begin{aligned}
&\sum_{j=1}^{n} \lambda_j^{1''} X_j \le \frac{\gamma_1^*}{\delta} X \\
&\sum_{j=1}^{n} \lambda_j^{1''} Z_j = \frac{\alpha^*}{\delta} Z \\
&\sum_{j=1}^{n} \lambda_j^{2''} Z_j = \frac{\alpha^*}{\delta} Z \\
&\sum_{j=1}^{n} \lambda_j^{2''} W_j \le \frac{\gamma_1^*}{\delta} W \\
&\sum_{j=1}^{n} \lambda_j^{2''} Y_j \ge \frac{\gamma_2^*}{\delta} Y \\
&\lambda''^{k} \ge 0, 1\lambda''^{k} = 1 \qquad k = 1,2.
\end{aligned} \tag{12}
$$

where $\lambda_j^{k''} = \frac{1}{\delta} \lambda_j^{k^*} \quad k = 1,2 \quad j = 1,\dots,n$.

If $\gamma_1^* = 1, \gamma_2^* > 1$ then $\frac{\gamma_2^*}{\delta} > \frac{\gamma_1^*}{\delta}$. In addition we have $(\frac{\gamma_1^*}{\delta} X, \frac{\alpha^*}{\delta} Z, \frac{\gamma_1^*}{\delta} W, \frac{\gamma_2^*}{\delta} Y) \in T_{NV}$.

Thus, (X, Z, W, Y) is not MPSS. Similar result is obtained by assumption $\gamma_1^* < 1$.

If $(X,Z,W,Y) \in T_{NV}$ is not MPSS, then there exists $(\alpha_1 X, \eta Z, \alpha_1 W, \beta Y) \in T_{NV}$ satisfying $\beta > \alpha_1$. Therefore we have

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$$\sum_{j=1}^{n}\lambda_j^1 X_j \le \alpha_1 X$$
$$\sum_{j=1}^{n}\lambda_j^1 Z_j = \eta Z$$
$$\sum_{j=1}^{n}\lambda_j^2 Z_j = \eta Z \tag{13}$$
$$\sum_{j=1}^{n}\lambda_j^2 W_j \le \alpha_1 W$$
$$\sum_{j=1}^{n}\lambda_j^2 Y_j \ge \beta Y$$
$$\lambda^k \ge 0\ ,\ 1\lambda^k = 1 \qquad k = 1,2.$$

Dividing above constraints by β results in the following:

$$\sum_{j=1}^{n}\hat{\lambda}_j^1 X_j \le \frac{\alpha_1}{\beta} X$$
$$\sum_{j=1}^{n}\hat{\lambda}_j^1 Z_j = \frac{\eta}{\beta} Z$$
$$\sum_{j=1}^{n}\hat{\lambda}_j^2 Z_j = \frac{\eta}{\beta} Z$$
$$\sum_{j=1}^{n}\hat{\lambda}_j^2 W_j \le \frac{\alpha_1}{\beta} W \tag{14}$$
$$\sum_{j=1}^{n}\hat{\lambda}_j^2 Y_j \ge Y$$
$$\sum_{j=1}^{n}\hat{\lambda}_j^1 = \sum_{j=1}^{n}\hat{\lambda}_j^2 = \frac{1}{\beta}$$
$$\hat{\lambda}^k \ge 0 \qquad k = 1,2.$$

where $\hat{\lambda}_j^k = \frac{1}{\beta}\lambda_j^k \quad k = 1,2 \quad j = 1,...,n$. If $\beta > \alpha_1$, we can find a solution for model (10) with $\gamma_1^* = \frac{\alpha_1}{\beta} < 1, \gamma_2^* = 1$.■

By using vectors $S^{-1} = (s_1^{-1},...,s_m^{-1}), S^{-2} = (s_1^{-2},...,s_q^{-2}), S^{+} = (s_1^{+},...,s_s^{+})$ model (10) becomes as follows:

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$$
\begin{aligned}
& Min \quad \frac{\gamma_1}{\gamma_2} - \varepsilon(1\mathrm{S}^{-1} + 1\mathrm{S}^{-2} + 1\mathrm{S}^{+}) \\
& s.t. \\
& \quad \sum_{j=1}^{n} \lambda_j^1 X_j + \mathrm{S}^{-1} = \gamma_1 X_o \\
& \quad \sum_{j=1}^{n} \lambda_j^1 Z_j = \alpha Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j^2 Z_j = \alpha Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j^2 W_j + \mathrm{S}^{-2} = \gamma_1 W_o \\
& \quad \sum_{j=1}^{n} \lambda_j^2 Y_j - \mathrm{S}^{+} = \gamma_2 Y_o \\
& \quad \sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 \\
& \quad \lambda^k \geq 0 \qquad\qquad k = 1,2 \\
& \quad \alpha > 0, \gamma_1 \leq 1, \gamma_2 \geq 1, \mathrm{S}^{-1} \geq 0, \mathrm{S}^{-2} \geq 0, \mathrm{S}^{+} \geq 0.
\end{aligned}
\tag{15}
$$

Dividing above constraints by γ_2 results in the following model:

$$
\begin{aligned}
& Min \quad \omega - \varepsilon(1\mathrm{S}'^{-1} + 1\mathrm{S}'^{-2} + 1\mathrm{S}'^{+}) \\
& s.t. \\
& \quad \sum_{j=1}^{n} \lambda_j'^1 X_j + \mathrm{S}'^{-1} = \omega X_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^1 Z_j = \alpha' Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^2 Z_j = \alpha' Z_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^2 W_j + \mathrm{S}'^{-2} = \omega W_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^2 Y_j - \mathrm{S}'^{+} = Y_o \\
& \quad \sum_{j=1}^{n} \lambda_j'^1 = \sum_{j=1}^{n} \lambda_j'^2 \\
& \quad \lambda'^k \geq 0 \qquad\qquad k = 1,2 \\
& \quad \mathrm{S}'^{-1} \geq 0, \mathrm{S}'^{-2} \geq 0, \mathrm{S}'^{+} \geq 0.
\end{aligned}
\tag{16}
$$

where $\lambda_j'^k = \frac{1}{\gamma_2}\lambda_j^k \quad k = 1,2, \omega = \frac{\gamma_1}{\gamma_2}, \alpha' = \frac{1}{\gamma_2}\alpha, \ \mathrm{S}'^{-1} = \frac{1}{\gamma_2}\mathrm{S}^{-1}, \mathrm{S}'^{-2} = \frac{1}{\gamma_2}\mathrm{S}^{-2}, \ \mathrm{S}'^{+} = \frac{1}{\gamma_2}\mathrm{S}^{+}$.

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Theorem9. Let $\frac{1}{\delta}(\gamma_1^* X_o - S^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - S^{-2^*}, \gamma_2^* Y_o + S^{+^*})$ be an optimal solution of model (15) and $\sum_{j=1}^{n} \lambda_j^{k^*} = \delta$.

Production possibility $\frac{1}{\delta}(\gamma_1^* X_o - S^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - S^{-2^*}, \gamma_2^* Y_o + S^{+^*})$ is MPSS.

Proof. 1) $\frac{1}{\delta}(\gamma_1^* X_o - S^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - S^{-2^*}, \gamma_2^* Y_o + S^{+^*}) \in T_{NV}$.

2) If $\frac{1}{\delta}(\gamma_1^* X_o - S^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - S^{-2^*}, \gamma_2^* Y_o + S^{+^*})$ is not MPSS, then there exists $\frac{1}{\delta}(\alpha_1(\gamma_1^* X_o - S^{-1^*}), \eta(\alpha^* Z_o), \alpha_1(\gamma_1^* W_o - S^{-2^*}), \beta(\gamma_2^* Y_o + S^{+^*})) \in T_{NV}$ satisfying $\beta > \alpha_1$. Therefore we have

$$
\begin{aligned}
&\sum_{j=1}^{n} \lambda_j^1 X_j \le \frac{1}{\delta}(\alpha_1(\gamma_1^* X_o - S^{-1*})) \\
&\sum_{j=1}^{n} \lambda_j^1 Z_j = \frac{1}{\delta}(\eta(\alpha^* Z_o)) \\
&\sum_{j=1}^{n} \lambda_j^2 Z_j = \frac{1}{\delta}(\eta(\alpha^* Z_o)) \\
&\sum_{j=1}^{n} \lambda_j^2 W_j \le \frac{1}{\delta}(\alpha_1(\gamma_1^* W_o - S^{-2*})) \\
&\sum_{j=1}^{n} \lambda_j^2 Y_j \ge \frac{1}{\delta}(\beta(\gamma_2^* Y_o + S^{+*})) \\
&\lambda^k \ge 0 \,,\ 1\lambda^k = 1 \qquad\qquad k = 1,2.
\end{aligned}
\tag{17}
$$

Dividing above constraints by $\frac{1}{\delta}\beta$ and using vectors

$\dot{S}^{-1} = (\dot{s}_1^{-1}, \ldots, \dot{s}_m^{-1}), \dot{S}^{-2} = (\dot{s}_1^{-2}, \ldots, \dot{s}_q^{-2}), \dot{S}^{+} = (\dot{s}_1^{+}, \ldots, \dot{s}_s^{+})$ result in the following:

$$
\begin{aligned}
&\sum_{j=1}^{n} \dot{\lambda}_j^1 X_j + \dot{S}^{-1} = \frac{\alpha_1}{\beta}(\gamma_1^* X_o) \\
&\sum_{j=1}^{n} \dot{\lambda}_j^1 Z_j = \dot{\alpha} Z_o \\
&\sum_{j=1}^{n} \dot{\lambda}_j^2 Z_j = \dot{\alpha} Z_o \\
&\sum_{j=1}^{n} \dot{\lambda}_j^2 W_j + \dot{S}^{-2} = \frac{\alpha_1}{\beta}(\gamma_1^* W_o) \\
&\sum_{j=1}^{n} \dot{\lambda}_j^2 Y_j - \dot{S}^{+} = \gamma_2^* Y_o \\
&\sum_{j=1}^{n} \dot{\lambda}_j^k = \frac{\delta}{\beta} \qquad\qquad k = 1,2 \\
&\dot{\lambda}^k \ge 0 \qquad\qquad k = 1,2.
\end{aligned}
\tag{18}
$$

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where $\dot{\lambda}_j^k = \frac{\delta}{\beta}\lambda_j^k \quad k=1,2 \quad j=1,...,n$. Now if $\alpha_1 < \beta$, we can find a solution for model (15) with $\frac{\alpha_1}{\beta}\gamma_1^* < \gamma_1^*$ which contradicts with the optimality of γ_1^*.■

Theorem9 gives a MPSS project for (X_o, Z_o, W_o, Y_o). Similar to the proof of theorem2, it can be easily proved that $(\gamma_1^* X_o - \mathrm{S}^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - \mathrm{S}^{-2^*}, \gamma_2^* Y_o + \mathrm{S}^{+^*})$ is efficient according to definition3. Therefore, $\frac{1}{\delta}(\gamma_1^* X_o - \mathrm{S}^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - \mathrm{S}^{-2^*}, \gamma_2^* Y_o + \mathrm{S}^{+^*}) \in T_{NV}$ is called a scale efficient target of (X_o, Z_o, W_o, Y_o).

Model (15) may have more than one optimal solution. Let $\{\lambda^k\}_{k=1}^2$ satisfy the following constraints:

$$\begin{aligned}
&\sum_{j=1}^n \lambda_j^1 X_j = \gamma_1^* X_o - \mathrm{S}^{-1*} \\
&\sum_{j=1}^n \lambda_j^1 Z_j = \alpha^* Z_o \\
&\sum_{j=1}^n \lambda_j^2 Z_j = \alpha^* Z_o \\
&\sum_{j=1}^n \lambda_j^2 W_j = \gamma_1^* W_o - \mathrm{S}^{-2*} \\
&\sum_{j=1}^n \lambda_j^2 Y_j = \gamma_2^* Y_o + \mathrm{S}^{+*} \\
&\sum_{j=1}^n \lambda_j^1 = \sum_{j=1}^n \lambda_j^2 \\
&\lambda^k \geq 0 \qquad\qquad k = 1,2 \\
&\alpha^* > 0, \gamma_1^* \leq 1, \gamma_2^* \geq 1, \mathrm{S}^{-1*} \geq 0, \mathrm{S}^{-2*} \geq 0, \mathrm{S}^{+*} \geq 0.
\end{aligned} \tag{19}$$

Then let $\sum_{j=1}^n \lambda_j^k = \delta \quad k=1,2$ and $\tau = \frac{1}{\delta}$.

Definition10. Production possibility $\tau(\gamma_1^* X_o - \mathrm{S}^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - \mathrm{S}^{-2^*}, \gamma_2^* Y_o + \mathrm{S}^{+^*}) \in T_{NV}$ is the largest MPSS project of (X_o, Z_o, W_o, Y_o) iff for every $\bar{\tau} > \tau$ we have $\bar{\tau}(\gamma_1^* X_o - \mathrm{S}^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - \mathrm{S}^{-2^*}, \gamma_2^* Y_o + \mathrm{S}^{+^*}) \notin T_{NV}$.

The smallest MPSS project of (X_o, Z_o, W_o, Y_o) is defined similarly.

According to above definition, the largest MPSS project of (X_o, Z_o, W_o, Y_o) is $\frac{1}{\delta}(\gamma_1^* X_o - \mathrm{S}^{-1^*}, \alpha^* Z_o, \gamma_1^* W_o - \mathrm{S}^{-2^*}, \gamma_2^* Y_o + \mathrm{S}^{+^*}) \in T_{NV}$ with the minimum value of δ. Therefore, the largest MPSS project of (X_o, Z_o, W_o, Y_o), according to definition10, is obtained by solving the following model:

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$$
\begin{aligned}
& Max \quad \tau \\
& s.t. \sum_{j=1}^{n} \lambda_j^1 X_j = \tau(\gamma_1^* X_o - S^{-1*}) \\
& \sum_{j=1}^{n} \lambda_j^1 Z_j = \tau(\alpha^* Z_o) \\
& \sum_{j=1}^{n} \lambda_j^2 Z_j = \tau(\alpha^* Z_o) \\
& \sum_{j=1}^{n} \lambda_j^2 W_j = \tau(\gamma_1^* W_o - S^{-2*}) \\
& \sum_{j=1}^{n} \lambda_j^2 Y_j = \tau(\gamma_2^* Y_o + S^{+*}) \\
& \sum_{j=1}^{n} \lambda_j^k = 1 \qquad k = 1,2 \\
& \lambda^k \geq 0 \qquad k = 1,2.
\end{aligned}
\tag{20}
$$

The smallest MPSS project of (X_o, Z_o, W_o, Y_o) is obtained by solving above model in minimization case.

The largest and the smallest MPSS projects of (X_o, Z_o, W_o, Y_o) are shown by (X, Z, W, Y) and (X', Z', W', Y'), respectively, and are determined according to the following formulas:

$$X = \tau^*(\gamma_1^* X_o - S^{-1*}) \quad Z = \tau^*(\alpha^* Z_o) \quad W = \tau^*(\gamma_1^* W_o - S^{-2*}) \quad Y = \tau^*(\gamma_2^* Y_o + S^{+*}) \tag{21}$$

$$X' = \tau'(\gamma_1^* X_o - S^{-1*}) \quad Z' = \tau'(\alpha^* Z_o) \quad W' = \tau'(\gamma_1^* W_o - S^{-2*}) \quad Y' = \tau'(\gamma_2^* Y_o + S^{+*}) \tag{22}$$

τ^* and τ' are the maximum and the minimum values of τ subject to the constraints of model (20). Dividing constraints of model (20) by γ_2^* results in the following:

$$
\begin{aligned}
& \sum_{j=1}^{n} \breve{\lambda}_j^1 X_j = \tau(\omega^* X_o - S'^{-1*}) \\
& \sum_{j=1}^{n} \breve{\lambda}_j^1 Z_j = \tau(\alpha'^* Z_o) \\
& \sum_{j=1}^{n} \breve{\lambda}_j^2 Z_j = \tau(\alpha'^* Z_o) \\
& \sum_{j=1}^{n} \breve{\lambda}_j^2 W_j = \tau(\omega^* W_o - S'^{-2*}) \\
& \sum_{j=1}^{n} \breve{\lambda}_j^2 Y_j = \tau(Y_o + S'^{+*}) \\
& \sum_{j=1}^{n} \breve{\lambda}_j^k = \frac{1}{\gamma_2^*} \qquad k = 1,2 \\
& \breve{\lambda}^k \geq 0 \qquad k = 1,2.
\end{aligned}
\tag{23}
$$

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where $\breve{\lambda}_j^k = \frac{1}{\gamma_2^*}\lambda_j^k, \alpha'^* = \frac{1}{\gamma_2^*}\alpha^* \quad k = 1,2$. Thus, the values of ω^*, α'^*, $S'^{-1^*}, S'^{-2^*}, S'^{+^*}$ are determined by solving model (16) and then are used to solve model (21). γ_2^* is obtained by solving model (21) (subject to the constraints (23)) and finally the value of γ_1^* is determined by using ω^* and γ_2^*.

2.3 Interval data

In imprecise data envelopment analysis (IDEA) the data can be interval e.g. when the missing values are replaced by intervals in which the unknown values are likely to belong. In interval efficiency assessment the final efficiency score for each DMU is characterized by an interval.

In the discussion to follow, we suppose that the imprecise data takes the forms of bounded data as follows:

$$\underline{y}_{rj} \le y_{rj} \le \overline{y}_{rj} \qquad \underline{x}_{ij} \le x_{ij} \le \overline{x}_{ij} \qquad \underline{z}_{kj} \le z_{kj} \le \overline{z}_{kj} \qquad r \in BO, i \in BI, k \in B. \tag{24}$$

where $\overline{y}_{rj}$, $\overline{x}_{ij}$ and $\overline{z}_{kj}$ and are the upper bounds and $\underline{y}_{rj}$, $\underline{x}_{ij}$ and $\underline{z}_{kj}$ are the lower bounds, and BO, BI and B represent the associated sets for bounded outputs , bounded inputs and bounded intermediate products respectively. The lower and upper bounds are assumed to be constants and strictly positive. According to model (4), the upper and lower bounds of the interval efficiency of DMU_o are obtained from the pessimistic and optimistic viewpoints, respectively, using models (25) and (26).

In the optimistic viewpoint the levels of inputs and outputs are adjusted in favor of the evaluated DMU_o and aggressively against the other units. In the pessimistic viewpoint, the levels of inputs and outputs are now adjusted unfavorably for under evaluation unit and in favor of the other units. Additionally, the intermediate product is the output of stage1 and the input of stage2; therefore, for evaluating whole process of DMU_o it cannot be considered in its lower or upper bound. We consider $\hat{z}_{kj} = \frac{1}{2}(\overline{z}_{kj} + \underline{z}_{kj}) \quad k \in B$.

Similarly, According to model (8), the upper and lower bounds of the interval efficiency of DMU_o are determined from the pessimistic and optimistic viewpoints.

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$$
\begin{aligned}
& Min\ \ \omega^{l} \\
& s.t. \\
& \sum_{j\neq o}\lambda'^{1}_{j}x^{l}_{ij}+\lambda'^{1}_{o}x^{u}_{io}\leq\omega^{l}x^{u}_{io} \qquad i\in BI \\
& \sum_{j=1}^{n}\lambda'^{1}_{j}x_{ij}\leq\omega^{l}x_{io} \qquad i\notin BI \\
& \sum_{j=1}^{n}\lambda'^{1}_{j}\hat{z}_{kj}=\omega^{l}_{1}\hat{z}_{ko} \qquad k\in B \\
& \sum_{j=1}^{n}\lambda'^{2}_{j}\hat{z}_{kj}=\omega^{l}_{1}\hat{z}_{ko} \qquad k\in B \\
& \sum_{j=1}^{n}\lambda'^{1}_{j}z_{kj}=\omega^{l}_{1}z_{ko} \qquad k\notin B \\
& \sum_{j=1}^{n}\lambda'^{2}_{j}z_{kj}=\omega^{l}_{1}z_{ko} \qquad k\notin B \\
& \sum_{j=1}^{n}\lambda'^{2}_{j}W_{j}\leq\omega^{l}W_{o} \\
& \sum_{j\neq o}\lambda'^{2}_{j}y^{u}_{rj}+\lambda'^{2}_{o}y^{l}_{ro}\geq y^{l}_{ro} \qquad r\in BO \\
& \sum_{j=1}^{n}\lambda'^{2}_{j}y_{rj}\geq y_{ro} \qquad r\notin BO \\
& \omega^{l}\leq\omega^{l}_{1},\ \omega^{l}_{1}\leq 1 \\
& \lambda'^{k}\geq 0 \qquad k=1,2.
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
& Min\ \ \omega^{u} \\
& s.t. \\
& \sum_{j\neq o}\lambda''^{1}_{j}x^{u}_{ij}+\lambda''^{1}_{o}x^{l}_{io}\leq\omega^{u}x^{l}_{io} \qquad i\in BI \\
& \sum_{j=1}^{n}\lambda''^{1}_{j}x_{ij}\leq\omega^{u}x_{io} \qquad i\notin BI \\
& \sum_{j=1}^{n}\lambda''^{1}_{j}\hat{z}_{kj}=\omega^{u}_{1}\hat{z}_{ko} \qquad k\in B \\
& \sum_{j=1}^{n}\lambda''^{2}_{j}\hat{z}_{kj}=\omega^{u}_{1}\hat{z}_{ko} \qquad k\in B \\
& \sum_{j=1}^{n}\lambda''^{1}_{j}z_{kj}=\omega^{u}_{1}z_{ko} \qquad k\notin B \\
& \sum_{j=1}^{n}\lambda''^{2}_{j}z_{kj}=\omega^{u}_{1}z_{ko} \qquad k\notin B \\
& \sum_{j=1}^{n}\lambda''^{2}_{j}W_{j}\leq\omega^{u}W_{o} \\
& \sum_{j\neq o}\lambda''^{2}_{j}y^{l}_{rj}+\lambda''^{2}_{o}y^{u}_{ro}\geq y^{u}_{ro} \qquad r\in BO \\
& \sum_{j=1}^{n}\lambda''^{2}_{j}y_{rj}\geq y_{ro} \qquad r\notin BO \\
& \omega^{u}\leq\omega^{u}_{1},\ \omega^{u}_{1}\leq 1 \\
& \lambda''^{k}\geq 0 \qquad k=1,2.
\end{aligned}
\tag{26}
$$

The largest MPSS projects of DMU_o in pessimistic and optimistic viewpoints are obtained by using model (20) with constraints (23) as follows:

$$
\begin{aligned}
& Max\ \ \tau^{l} \\
& s.t. \\
& \sum_{j\neq o}\breve{\lambda}^{1}_{j}x^{l}_{ij}+\breve{\lambda}^{1}_{o}x^{u}_{io}=\tau^{l}(\omega^{*l}x^{u}_{io}-s'^{-1*}_{i}) \qquad i\in BI \\
& \sum_{j=1}^{n}\breve{\lambda}^{1}_{j}x_{ij}=\tau^{l}(\omega^{*l}x_{io}-s'^{-1*}_{i}) \qquad i\notin BI \\
& \sum_{j=1}^{n}\breve{\lambda}^{1}_{j}\hat{z}_{kj}=\tau^{l}(\alpha'^{*}\hat{z}_{ko}) \qquad k\in B \\
& \sum_{j=1}^{n}\breve{\lambda}^{2}_{j}\hat{z}_{kj}=\tau^{l}(\alpha'^{*}\hat{z}_{ko}) \qquad k\in B
\end{aligned}
\tag{27}
$$

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$$\sum_{j=1}^{n} \breve{\lambda}_j^1 z_{kj} = \tau^l (\alpha'^* z_{ko}) \qquad k \notin B$$

$$\sum_{j=1}^{n} \breve{\lambda}_j^2 z_{kj} = \tau^l (\alpha'^* z_{ko}) \qquad k \notin B$$

$$\sum_{j=1}^{n} \breve{\lambda}_j^2 W_j = \tau^l (\omega^{*l} W_o - \mathrm{S}'^{-2*})$$

$$\sum_{j \neq o} \breve{\lambda}_j^2 y_{rj}^u + \breve{\lambda}_o^2 y_{ro}^l = \tau^l (y_{ro}^l + s_r'^{+*}) \qquad r \in BO$$

$$\sum_{j=1}^{n} \breve{\lambda}_j^2 y_{rj} = \tau^l (y_{ro} + s_r'^{+*l}) \qquad r \notin BO$$

$$\sum_{j=1}^{n} \breve{\lambda}_j^k = \frac{1}{\gamma_2^{*l}} \qquad k = 1,2$$

$$\breve{\lambda}^k \geq 0 \qquad k = 1,2.$$

$Max \quad \tau^u$

$s.t.$

$$\sum_{j \neq o} \widehat{\lambda}_j^1 x_{ij}^u + \widehat{\lambda}_o^1 x_{io}^l = \tau^u (\omega^{*u} x_{io}^l - s_i''^{-1*}) \qquad i \in BI$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^1 x_{ij} = \tau^u (\omega^{*u} x_{io} - s_i''^{-1*}) \qquad i \notin BI \tag{28}$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^1 \hat{z}_{kj} = \tau^u (\alpha''^* \hat{z}_{ko}) \qquad k \in B$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^2 \hat{z}_{kj} = \tau^u (\alpha''^* \hat{z}_{ko}) \qquad k \in B$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^1 z_{kj} = \tau^u (\alpha''^* z_{ko}) \qquad k \notin B$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^2 z_{kj} = \tau^u (\alpha''^* z_{ko}) \qquad k \notin B$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^2 W_j = \tau^u (\omega^{*u} W_o - \mathrm{S}''^{-2*})$$

$$\sum_{j \neq o} \widehat{\lambda}_j^2 y_{rj}^u + \widehat{\lambda}_o^2 y_{ro}^l = \tau^u (y_{ro}^l + s_r''^{+*}) \qquad r \in BO$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^2 y_{rj} = \tau^u (y_{ro} + s_r''^{+*}) \qquad r \notin BO$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^k = \frac{1}{\gamma_2^{*u}} \qquad k = 1,2$$

$$\widehat{\lambda}^k \geq 0 \qquad k = 1,2.$$

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The values of $\omega^{*l}, s'^{-1^*}, s'^{-2^*}, s'^{+^*}, \alpha'^*, \omega^{*u}, s''^{-1^*}, s''^{-2^*}, s''^{+^*}, \alpha''^*$ are determined by solving model (16) from the pessimistic and optimistic viewpoints, respectively.

3 Numerical example

Our data, which is shown in Table 1, is drawn from 23 periods (monthly) of some metal can making factory production. Inputs to the first stage are tinplate (x_1), lacquer (x_2) and working days (x_3). In this stage, the tin plates are cut out to standard sizes and coated with lacquer. The intermediate output of the first stage is the coated tin plates with lacquer (z). In the second stage, coated tin plates, produced in the first stage, seam welding metal powder (w_1), seam welding wire (w_2) and liquid rubber (w_3) are used to produce the cans bodies and ends. Cans (y) are the outputs of stage 2. The output data of the second stage is based on the total number of produced cans bodies and ends. We assume each period of production as a DMU. The total number of cans is determined and the required raw materials are predicted as interval values.

Table 1 Data

	x_{j1} *Tn*	x_{j2} *Kg*	x_{j3}	z_j *Tn*	w_{j1} *Kg*	w_{j2} *Kg*	w_{j3} *Kg*	y
1	[1900,2990]	[200,350]	25	[1680,1850]	8070	40040	21935	15905961
2	9270	[650,1050]	24	7100	34270	170010	102935	70861554
3	[2700,3200]	590	29	[2100,2300]	10050	55010	25720	28663362
4	[12200,13850]	2250	28	12000	80730	320040	195280	106565811
5	10500	[1900,2200]	28	[9100,9500]	54090	220980	125920	100458404
6	[8200,8600]	1920	20	7700	90690	180040	98320	85825021
7	[1385,1480]	170	23	1250	3930	21490	11190	14139320
8	15800	3350	28	13400	95210	400700	160055	110921244
9	[1530,2400]	270	29	[1100,1190]	4970	19030	11230	14044097
10	18030	[4500,5200]	26	15090	115710	470040	280590	85450325
11	[17930,19000]	[5470,6590]	27	[16000,17000]	90000	250930	190800	56623490
12	14560	1640	21	11100	85000	180015	85150	80261500
13	[2990,3110]	[610,690]	20	[2200,2500]	11200	55010	22900	26090450
14	[17300,20300]	7250	27	17٠00	120500	580050	360500	93155023
15	3330	380	28	[1900,2100]	8200	40970	20950	22400750
16	9880	[1600,2010]	27	8500	42000	180050	78320	90950236
17	[5580,6020]	630	17	5395	20500	100540	60880	76310122
18	18500	[3550,3900]	27	16100	113800	180310	150000	68365390
19	7000	[920,1100]	26	[5500,6500]	٨0000	140040	86300	80844920
20	15700	[1500,2200]	25	[11800,13000]	100500	250060	147700	90662025
21	1860	210	19	[800,900]	40150	34070	20600	14100580
22	[310,400]	45	17	295	1500	7000	4700	4520675
23	[700,920]	[60,105]	20	680	3400	16690	10100	11290423

Table 2 reports the results of models (25) and (26), pessimistic and optimistic scores based on model (8) and tables 3 does the results of models (27) and (28), respectively.

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Table 2 Results of models (25) and (26), pessimistic and optimistic scores based on model (8)

	$[\omega^l, \omega^u]$	$[\rho^l, \rho^u]$
1	[0.5755,0.7883]	[0.4789,0.5608]
2	[0.6465,0.7742]	[0.5820,0.6487]
3	[0.9014,0.9182]	[0.6509,0.6813]
4	[0.8035,0.8134]	[0.5046,0.5391]
5	[0.7662,0.7814]	[0.6025,0.6373]
6	[0.9186,0.9487]	[0.6968,0.7349]
7	[0.9310,1]	[0.8355,1]
8	[0.7790,0.8006]	[0.4854,0.5023]
9	[0.7462,0.8897]	[0.6542,0.6986]
10	[0.6955,0.7452]	[0.3475,0.3599]
11	[0.5429,0.5645]	[0.3010,0.3097]
12	[0.8902,1]	[0.5004,0.7107]
13	[0.8997,1]	[0.5986,0.8238]
14	[0.6677,0.6911]	[0.3404,0.3536]
15	[0.8472,0.8580]	[0.6176,0.6254]
16	[0.8624,0.9179]	[0.6640,0.7110]
17	[1,1]	[1,1]
18	[0.6697,0.6925]	[0.3880,0.4001]
19	[0.8408,1]	[0.7277,0.7731]
20	[0.9542,0.9582]	[0.5286,0.5286]
21	[0.7356,0.9539]	[0.5646,0.7705]
22	[0.9048,0.9422]	[0.7214,0.7690]
23	[0.8821,1]	[0.7879,1]

Table 3 Results of models (27) and (28)

	γ_1^{*l}	γ_2^{*l}	τ^{*l}	γ_1^{*u}	γ_2^{*u}	τ^{*u}
1	0.7290	1.2790	3.7304	0.8761	1.1346	1.4037
2	0.8914	1.7741	0.6018	1	1.2990	0.8333
3	0.9000	1	2.6622	0.9167	1	0.8333
4	0.7811	1	0.6850	0.8100	1	0.7000
5	0.7973	1.1945	0.6939	0.9100	1.0945	0.7091
6	0.9476	1.1288	0.7866	0.9500	1.1160	0.9700
7	0.9293	1	5.3970	1	1	1
8	0.7500	1	0.6596	0.7913	1	0.7321
9	0.7369	1	5.4336	0.7941	1	1.1969
10	0.6841	1	0.9461	0.7325	1	0.8696
11	0.5321	1.3100	1.1538	0.6115	1.1310	0.8000
12	0.8550	1	0.8010	1	1	1
13	0.9370	1.0913	2.6597	1	1	1
14	0.8500	1.3000	0.7160	0.8600	1.2533	0.6131
15	0.6714	1.1569	2.9400	0.8718	1.0416	0.9091
16	0.8470	1.0416	0.8021	0.9007	1	0.8000
17	1	1	1	1	1	1
18	0.5718	1.0221	1.100	0.6723	1	1.1934
19	0.9700	1.1609	0.8049	1	1	1
20	0.9511	1	0.8471	0.9561	1	0.9024
21	0.9500	1.3760	3.0137	0.9531	1	1
22	0.9546	1.1476	14.6302	1	1.0600	4.6970
23	1	1.1127	6.0710	1	1	1

According to the results of table 2, DMU17 (period 17) is efficient in the cases of optimistic and pessimistic viewpoints.

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In the case of optimistic viewpoint, for DMUs 7, 12, 13, 17, 19 and 23 we have $\gamma_1^{*u}=1, \gamma_2^{*u}=1$. Therefore, these DMUs are MPSS according to theorem8.

Additionally, for projecting DMUs 3 to 5, 8, 10, 11, 14, 15, 16, 18 and 20(periods 2 to 5, 8, 10, 11, 14, 15, 16, 18 and 20) into the largest MPSS, respectively, firstly, all inputs and output values have been scaled so that the inputs and the output values have been decreased because of the inequalities $\gamma_1^{*u}\tau^{*u}<1$ and $\gamma_2^{*u}\tau^{*u}<1$. Then, the inputs and outputs values have been adjusted, to project DMUs on the largest MPSS, by suitable values of slacks.

The results from the pessimistic viewpoint can be interpreted as above argue.

Results of formula (21) in the cases of pessimistic and optimistic viewpoints are shown in table4. All DMUs are projected to their positions.

Table 4 Results of formula (21) in the cases of pessimistic and optimistic viewpoints

	$\dot{x}_1$	$\dot{x}_2$	$\dot{x}_3$	$\dot{w}_1$	$\dot{w}_2$	$\dot{w}_3$	$\dot{y}$	$\ddot{x}_1$	$\ddot{x}_2$	$\ddot{x}_3$	$\ddot{w}_1$	$\ddot{w}_2$	$\ddot{w}_3$	$\ddot{y}$
1	5580	630	17	20500	100540	60880	76310122	1525.8713	186.6207	23	8669.2943	31788.7027	19266.8970	2.53364E+7
2	5580	630	17	20500	100540	60880	76310122	3352.5727	541.6667	20	28556.3333	96545.8333	59652.5833	6.44513E+7
3	5580	630	17	20500	100540	60880	76310122	1122.5106	192.0463	17	8375.0000	24808.5000	15070.9167	2.38862E+7
4	5580	630	17	20500	100540	60880	76310122	5580	630	17	20500	100540	60880	7.63101E+7
5	5580	630	17	20500	100540	60880	76310122	5580	630	17	20500	100540	60880	7.63101E+7
6	5580	630	17	20500	100540	60880	76310122	7000	920	26	90000	140040	86300	8.08449E+7
7	5580	630	17	20500	100540	60880	76310122	1385	170	23	3930	21490	11190	14139320
8	5580	630	17	20500	100540	60880	76310122	7000	920	26	90000	140040	86300	8.08449E+7
9	5580	630	17	20500	100540	60880	76310122	1385	170	23	3930	21490	11190	14139320
10	5580	630	17	20500	100540	60880	76310122	5580	630	17	20500	100540	60880	7.63101E+7
11	5580	630	17	20500	100540	60880	76310122	5580	630	17	20500	100540	60880	7.63101E+7
12	5580	630	17	20500	100540	60880	76310122	14560	4640	21	85000	180015	85150	8.02615E+7
13	5580	630	17	20500	100540	60880	76310122	2990	610	20	11200	55010	22900	2.60904E+7
14	5580	630	17	20500	100540	60880	76310122	5580	630	17	20500	100540	60880	7.63101E+7
15	5580	630	17	20500	100540	60880	76310122	1113.8400	190.1962	17	7454.5455	24267.1818	14741.8182	2.13644E+7
16	5580	630	17	20500	100540	60880	76310122	5580	630	17	20500	100540	60880	7.63101E+7
17	6020	630	17	20500	100540	60880	76310122	5580	630	17	20500	100540	60880	7.63101E+7
18	5580	630	17	20500	100540	60880	76310122	14560	4640	21	85000	180015	85150	8.02615E+7
19	5580	630	17	20500	100540	60880	76310122	7000	920	26	90000	140040	86300	8.08449E+7
20	5580	630	17	20500	100540	60880	76310122	7000	920	26	90000	140040	86300	8.08449E+7
21	5580	630	17	20500	100540	60880	76310122	1385	170	23	3930	21490	11190	14139320
22	5580	630	17	20500	100540	60880	76310122	934.9982	110.5756	22	6196.4816	18009.5587	10897.1985	2.22336E+7
23	5580	630	17	20500	100540	60880	76310122	700	60	20	3400	16690	10100	1.12904E+7

4 Conclusion

In this paper we have considered the production systems in which inputs in the first stage produce intermediate outputs transformed in the second stage to the final outputs. The second stage, in addition to intermediate flows from the first stage, may have its own inputs. This paper addresses units with such two-stage structure and models have been proposed to measure efficiency scores of this type of two-stage structure DMUs. One of the most important concepts about these systems is identifying the most productive scale size (MPSS) pattern that can help management to identify the future improvement for DMUs. This paper presents models to project two-stage DMU with interval data on the scale efficient targets correspond to the largest and the smallest MPSS projects.

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