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Output-input ratio analysis and data envelopment analysis inefficient frontier

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Abstract Data Envelopment Analysis (DEA) is a mathematical programming approach to evaluate the relative efficiency of decision-making units (DMUs) which use multiple inputs to produce multiple outputs. Identification of the DMUs forming the frontier before using DEA is of great importance to have an effective calculation. This article introduces the worst efficiency analysis approach in which an inefficient production frontier is used to determine the worst relative efficiency score that can be assigned to any DMU. Furthermore, mathematical properties determining the intrinsic relationships between the inefficient frontier DMUs and the output-input ratios are discussed. It was observed that a high ranked performance in the ratio analysis indicates a DEA frontier. This in turn allows the identification of membership of frontier DMUs without solving a DEA program. This finding is helpful in simplifying the DEA solution.

Keyword: Data envelopment analysis, efficiency, inefficient frontier, ratio analysis, returns to scale.

1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric method to evaluate the relative efficiency of decision-making units (DMUs) based on multiple inputs and outputs [1,2]. This approach is based on mathematical programming models [3-5]. Each DEA model is solved n times to analyze a set of DMUs, once for each target DMU. Therefore, this method is computationally costly. The concepts, allowing the effective calculation are based on computational methods for mathematical programming such as advanced foundations and candidate lists. As discussed by Ali [6, 7, 8], the computational structures simplifying DEA computations are required in order to shorten such time-consuming calculations. Accordingly, the identification of the frontier (or non-frontier) membership of DMUs without solving a DEA program has a great effect in simplifying program solving. Chen and Ali [9] discussed intrinsic relationships between input/output ratios and DMUs forming the efficient production frontier. In this article, mathematical characteristics related to the intrinsic relationship between input/output ratios and DMUs forming the inefficient production frontier are discussed. Accordingly, the membership of the frontier DMU can be determined before DEA calculation.

The article is organized as follows: Section 2 introduces the inefficient production frontier and basic models for worst efficiency analysis. These models have a structure similar

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to the CCR and BCC DEA models. Section 3 analyzes the output-input ratio based on the worst efficiency analysis models. The model is applied to the Iranian gas companies in Section 4. The conclusions are provided in Section 5.

2 Background

2.1 Basic models for analyzing the worst relative efficiency

Assume that there are n operating units whose performance must be measured. In these units, m inputs are used to produce s outputs. To comply with DEA terms, the term DMU is used for these operating units. However, some of these units may not have decision-making power. The observed values for inputs and outputs are respectively shown by x_{ij} ($i=1,\dots,m$) and y_{rj} ($r=1,\dots,s$) for DMU_j ($j=1,\dots,n$). Assuming that all inputs and outputs are positive, the efficiency of DMU_j is defined as follows:

$$\theta_j=\frac{\sum_{r=1}^{s}u_r y_{rj}}{\sum_{i=1}^{m}v_i x_{ij}} \tag{1}$$

Where u_r and v_i are the weights assigned to the output r and the input i. To determine the best possible efficiency score for a given DMU_o compared to the other DMUs, Charnes et al. [2] used the following model, later known as CCR model:

$$\begin{aligned}\max\quad & \theta_o=\frac{\sum_{r=1}^{s}u_r y_{ro}}{\sum_{i=1}^{m}v_i x_{io}}\\ \text{s.t.}\quad & \theta_j=\frac{\sum_{r=1}^{s}u_r y_{rj}}{\sum_{i=1}^{m}v_i x_{ij}}\le 1,\quad j=1,\dots,n,\\ & u_r, v_i\ge 0,\quad r=1,\dots,s;\quad i=1,\dots,m,\end{aligned} \tag{2}$$

This fractional programming model can be converted to the following linear programming (LP) model [10]:

$$\begin{aligned}\max\quad & \theta_o=\sum_{r=1}^{s}u_r y_{ro}\\ \text{s.t.}\quad & \sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}\le 0,\quad j=1,\dots,n,\\ & \sum_{i=1}^{m}v_i x_{io}=1,\\ & u_r, v_i\ge 0,\quad r=1,\dots,s;\quad i=1,\dots,m.\end{aligned} \tag{3}$$

If there is a set of positive weights to make an optimal value of 1 for θ_o ($\theta_o^*=1$), then DMU_o is called DEA-efficient or optimistic efficient. If $\theta_o^*<1$, the DMU_o will be an DEA-non-efficient. It should be noted that DEA-non-efficient does not necessarily mean DEA-

inefficient. In fact, DEA-efficient and DEA-inefficient are only two extremes. For n different DMUs, n LP models must be solved to produce n different sets of weights.
The model (4) is used to determine the worst possible efficiency score of a DMU relative to the best possible relative efficiency score determined by LP (3) [11-14]:

$$\begin{aligned} \min \quad & \phi_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ \text{s.t.} \quad & \phi_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \ge \alpha, \quad j = 1,\ldots,n, \\ & u_r, v_i \ge 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m, \end{aligned} \tag{4}$$

Where α ($\alpha > 0$) represents the minimum value of all scores of the worst possible relative efficiency.
Now, consider the following model [15-17]:

$$\begin{aligned} \min \quad & \phi_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ \text{s.t.} \quad & \phi_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \ge 1, \quad j = 1,\ldots,n, \\ & u_r, v_i \ge 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m, \end{aligned} \tag{5}$$

The optimum objective function value of the model (5) multiplied by α is equal to the optimum value of the objective function of the model (4). Applying the usual conversions to the fractional programming model (5), the following LP model is obtained:

$$\begin{aligned} \min \quad & \phi_o = \sum_{r=1}^{s} u_r y_{ro} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \ge 0, \quad j = 1,\ldots,n, \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r, v_i \ge 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m. \end{aligned} \tag{6}$$

For simplicity, the model (6) is rewritten in the vector form as follows:

$$\begin{aligned} & \min \mathbf{u}\mathbf{y}_o \\ & \text{s.t.} \\ & \mathbf{u}Y - \mathbf{v}X \ge \mathbf{0} \\ & \mathbf{v}\mathbf{x}_o = 1 \\ & \mathbf{u} = (u_r) \ge \mathbf{0} \\ & \mathbf{v} = (v_i) \ge \mathbf{0} \end{aligned} \tag{7}$$

Where $Y = (\mathbf{y}_i)$ and $X = (\mathbf{x}_i)$ respectively show the output and input matrices of all DMUs, and $\mathbf{y} = (y_{rj})$ and $\mathbf{x} = (x_{ij})$ represent the output and input vectors for DMU_j.

Definition 1: If there is a set of positive weights for which the optimal objective function value of the model (6) is equal to 1 ($\phi_o^* = 1$), then DMU_o is called pessimistic inefficient or DEA-inefficient.

Definition 2: If DMU_o is neither DEA-efficient nor DEA-inefficient, then it will be an DEA-unspecified.

In general, DEA-efficient DMUs are those with a good performance while DEA-inefficient DMUs are those with a bad performance. DEA-unspecified DMUs are those whose efficiencies are neither good nor bad. Therefore, DMUs with the best and worst efficiency are respectively identified as DEA-efficient and DEA-inefficient DMUs. All DEA-inefficient DMUs define an inefficient production frontier as a basis for the following definitions and assumptions:

Definition 3: The inefficient production possibility set is a set of

$$\hat{T} = \{(X, Y) \mid Y \geq 0 \text{ can be produced from } X \geq 0\} \tag{8}$$

The following assumptions define the inefficient production possibility set:

Assumption 1 (Convexity): If $(X_j, Y_j) \in \hat{T}$ $(j = 1, \ldots, n)$ and $\lambda_j \geq 0$ are non-negative scalars so that $\sum_{j=1}^{n} \lambda_j = 1$, then $(\sum_{j=1}^{n} \lambda_j X_j, \sum_{j=1}^{n} \lambda_j Y_j) \in \hat{T}$.

Assumption 2 (Disposability): (a) If $(X, Y) \in \hat{T}$ and $\bar{X} \leq X$, then $(\bar{X}, Y) \in \hat{T}$. (b) if $(X, Y) \in \hat{T}$ and $\bar{Y} \geq Y$, then $(X, \bar{Y}) \in \hat{T}$.

Assumption 3 (Ray unboundedness): If $(X, Y) \in \hat{T}$, then $(kX, kY) \in \hat{T}$ for $k \geq 0$.

Assumption 4 (Minimum extrapolation): $\hat{T}$ is a set of intersection of all T' in which the assumptions 1, 2, and 3 apply provided that $(X_j, Y_j) \in T'$ $(j = 1, \ldots, n)$ for any observed vector.

The inefficient production possibility set can be determined uniquely with the following relationship [15,18]:

$$\hat{T} = \left\{ (X, Y) \left| \sum_{j=1}^{n} \lambda_j X_j \geq X, \sum_{j=1}^{n} \lambda_j Y_j \leq Y, \lambda_j \geq 0, j = 1, \ldots, n \right. \right\} \tag{9}$$

The following dual for LP (6) facilitates the interpretation of inefficiency [13,19]:

$$\begin{aligned} & \max \phi_o \\ & \text{s.t.} \\ & \boldsymbol{\lambda} X \geq \phi_o \mathbf{x}_o \\ & \boldsymbol{\lambda} Y \leq \mathbf{y}_o \\ & \boldsymbol{\lambda} \geq \mathbf{0} \end{aligned} \tag{10}$$

Assume that the inputs are increased with the same ϕ_o ratio and the outputs are kept unchanged. If the input cannot be increased in the same ratio, i.e. $\phi_o^* = 1$, then DMU_o is a DEA-inefficient or pessimistic inefficient. On the other hand, if inputs can be increased with the same ratio, i.e. $\phi_o^* > 1$, then DMU_o is not DEA-inefficient. All DEA-inefficient DMUs form an inefficient frontier called the maximum input frontier. In contrast, the traditional production frontier can be considered the maximum output frontier. The models (6), (7), and (10) are the basic models for worst efficiency analysis. These models are quite similar to the CCR DEA models.

2.2 Technical efficiency models for analyzing the worst relative efficiency

Traditional DEA include the BCC model, additive DEA models as well as the CCR model. In addition to the basic CCR worst efficiency analysis model introduced in the previous section, a technical efficiency model with the same structure of the BCC model is introduced for analyzing the worst efficiency.

To measure the relative performance of a DMU in terms of technical efficiency, the following technical efficiency model can be used for analyzing the worst relative efficiency:

$$\begin{aligned} \min \quad & \phi_o = \frac{\sum_{r=1}^{s} u_r y_{ro} + u_o}{\sum_{i=1}^{m} v_i x_{io}} \\ \text{s.t.} \quad & \phi_j = \frac{\sum_{r=1}^{s} u_r y_{rj} + u_o}{\sum_{i=1}^{m} v_i x_{ij}} \geq 1, \quad j = 1, \ldots, n, \\ & u_r, v_i \geq 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m, \quad u_o \text{ free} \end{aligned} \tag{11}$$

The model (11) is converted to the following LP:

$$\begin{aligned} \min \quad & \phi_o = \sum_{r=1}^{s} u_r y_{ro} + u_o \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} + u_o - \sum_{i=1}^{m} v_i x_{ij} \geq 0, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r, v_i \geq 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m, \quad u_o \text{ free} \end{aligned} \tag{12}$$

This model is rewritten in the vector form as follows:

$$\begin{aligned} & \min \quad \mathbf{u}\mathbf{y}_o + u_o \\ & \text{s.t.} \\ & \mathbf{u}Y + u_o\mathbf{e} - \mathbf{v}X \geq \mathbf{0} \\ & \mathbf{v}\mathbf{x}_o = 1 \\ & \mathbf{u} = (u_r) \geq \mathbf{0} \\ & \mathbf{v} = (v_i) \geq \mathbf{0} \\ & u_o \text{ free} \end{aligned} \tag{13}$$

Where $\mathbf{e}$ is a n-dimensional vector all components of which equal unity. The LP (13) dual is as follows:

$$\begin{aligned} & \max \phi_o \\ & \text{s.t.} \\ & \boldsymbol{\lambda}X \geq \phi_o \mathbf{x}_o \\ & \boldsymbol{\lambda}Y \leq \mathbf{y}_o \\ & \mathbf{e}\boldsymbol{\lambda} = 1 \\ & \boldsymbol{\lambda} \geq \mathbf{0} \end{aligned} \tag{14}$$

The models (11)-(14) are technical efficiency models for worst efficiency analysis. These models are also called BCC worst efficiency analysis models because they are based on BCC DEA models.

Definition 4: If there is a set of positive weights for which the optimal objective function value in the models (12) or (13) is equal to unity ($\phi_o^* = 1$), then DMU_o is technically inefficient.

In the BCC worst efficiency analysis models, the assumptions 1, 2, and 4 are applied to the inefficient production possibility set $\hat{T}$ and are uniquely determined through the following relationship:

$$\hat{T} = \left\{ (X, Y) \middle| \sum_{j=1}^{n} \lambda_j X_j \ge X, \sum_{j=1}^{n} \lambda_j Y_j \le Y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0, j = 1, \ldots, n \right\} \tag{15}$$

Technical inefficiency can be interpreted in the same way as the use of the model (10) to interpret the inefficiency.

Further, the BCC worst efficiency analysis model is as follows in the case of the output oriented:

$$\begin{aligned} \max \quad & \varphi_o = \frac{\sum_{i=1}^{m} v_i x_{io} - v_o}{\sum_{r=1}^{s} u_r y_{ro}} \\ \text{s.t.} \quad & \varphi_j = \frac{\sum_{i=1}^{m} v_i x_{ij} - v_o}{\sum_{r=1}^{s} u_r y_{rj}} \le 1, \quad j = 1, \ldots, n, \\ & u_r, v_i \ge 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m, \quad u_o \text{ free} \end{aligned} \tag{16}$$

3 Frontier DMUs

3.1 Constant returns to scale (CRS)

Theorem 1: If there is a combination of the weights $\tilde{\upsilon}_i \ge 0$ $(i = 1, \ldots, m)$ and $\tilde{\mu}_r \ge 0$ $(r = 1, \ldots, s)$ so that

$$\frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rk}}{\sum_{i=1}^{m} \tilde{\upsilon}_i x_{ik}} = \min_j \left\{ \frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rj}}{\sum_{i=1}^{m} \tilde{\upsilon}_i x_{ij}} \right\} \tag{17}$$

Then the DMU_k is located on the CRS inefficient frontier.

Proof: For $\tilde{\upsilon}_i \ge 0$ and $\tilde{\mu}_r \ge 0$, the following symbol is used:

$$h_k = \min_j \left\{ \frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rj}}{\sum_{i=1}^{m} \tilde{\upsilon}_i x_{ij}} \right\} = \left(\frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rk}}{\sum_{i=1}^{m} \tilde{\upsilon}_i x_{ik}} \right) \tag{18}$$

Assume that $v_i = h_k \tilde{\upsilon}_i$ $(i = 1, \ldots, m)$ and $u_r = \tilde{\mu}_r$ $(r = 1, \ldots, s)$. Then for DMU_k, we have:

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} = \frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rj}}{\sum_{i=1}^{m} \tilde{\upsilon}_i x_{ij}} \Big/ h_k \ge 1, \quad j = 1, \ldots, n \tag{19}$$

Which satisfies the constraints of the model (5), and

$$\frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} = \frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rk}}{\sum_{i=1}^{m} \tilde{\upsilon}_i x_{ik}} \Big/ h_k = 1 \tag{20}$$

represents a value of 1 for the objective function in the model (5).

Therefore, DMU_k is located on the CRS inefficient frontier.

Theorem 1 provides the ability to identify existing DMUs on the CRS inefficient frontier, because such DMUs show the lowest integrated output ratio to the integrated inputs using some selected weights. Accordingly, this theorem allows for the identification of "frontier DMUs" through selecting different combinations of inputs and outputs to determine a particular set of weights.

Result 1: DMU_k is located on the CRS inefficient frontier if

$$\frac{\sum_{r\in R} y_{rk}}{\sum_{i\in I} x_{ik}} = \min_j \left\{ \frac{\sum_{r\in R} y_{rj}}{\sum_{i\in I} x_{ij}} \right\} \tag{21}$$

Where $I \subseteq \{1,\ldots,m\}$ is a subset of inputs and $R \subseteq \{1,\ldots,s\}$ is a subset of outputs.

Proof: Assume that $I \subseteq \{1,\ldots,m\}$ is a subset of inputs and $R \subseteq \{1,\ldots,s\}$ is a subset of outputs. Also, assume that $\tilde{\upsilon}_i = 1$ for $i \in I$ and $\tilde{\upsilon}_i = 0$ for $i \notin I$ and $\tilde{\mu}_r = 1$ for $r \in R$ and $\tilde{\mu}_r = 0$ for $r \notin R$ are given. Using the following symbol:

$$h_k = \min_j \left\{ \frac{\sum_{r\in R} y_{rj}}{\sum_{i\in I} x_{ij}} \right\} = \left(\frac{\sum_{r\in R} y_{rk}}{\sum_{i\in I} x_{ik}} \right) \tag{22}$$

Assume that $v_i = h_k$ for $i \in I$ and $\tilde{\upsilon}_i = 0$ for $i \notin I$ and $\tilde{\mu}_r = 1$ for $r \in R$ and $\tilde{\mu}_r = 0$ for $r \notin R$ are given. Then for DMU_k we have:

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} = \frac{\sum_{r\in R} y_{rj}}{h_k \sum_{i\in I} x_{ij}} \geq 1, \quad j = 1,\ldots,n \tag{23}$$

which satisfies the constraints in the model (5) and

$$\frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} = \frac{\sum_{r\in R} y_{rk}}{h_k \sum_{i\in I} x_{ik}} = 1 \tag{24}$$

represents a value of 1 for the objective function in the model (5).

Therefore, DMU_k is located on the CRS inefficient frontier.

When $I = \{1,\ldots,m\}$ and $R = \{1,\ldots,s\}$, it immediately leads to the following results:

Results 2: DMU_k is located on the CRS inefficient frontier if

$$\frac{\sum_{r=1}^{s} y_{rk}}{\sum_{i=1}^{m} x_{ik}} = \min_j \left\{ \frac{\sum_{r=1}^{s} y_{rj}}{\sum_{i=1}^{m} x_{ij}} \right\} \tag{25}$$

In addition, if I and R only have one component, then:

Results 3: DMU_k is located on the CRS inefficient frontier if

$$\frac{y_{rk}}{x_{ik}} = \min_j \left\{ \frac{y_{rj}}{x_{ij}} \right\}, \quad \text{for any } i = 1,\ldots,m \text{ and } r = 1,\ldots,s. \tag{26}$$

3.2 Variable returns to scale (VRS)

It is well established that a DMU that is on the CRS inefficient frontier is also on the VRS envelopment inefficient frontier. As a result, the theorem 1 and the results 1 to 3 can also be used for VRS. In addition, we have:

Theorem 2:

(i) If there is a combination of $\tilde{\upsilon}_i \geq 0$, $\tilde{\mu}_r \geq 0$, and $\tilde{\mu}_o$ weights so that

$$\frac{\sum_{r=1}^{s}\tilde{\mu}_r y_{rk} + \tilde{\mu}_o}{\sum_{i=1}^{m}\tilde{\upsilon}_i x_{ik}} = \min_j \left\{ \frac{\sum_{r=1}^{s}\tilde{\mu}_r y_{rj} + \tilde{\mu}_o}{\sum_{i=1}^{m}\tilde{\upsilon}_i x_{ij}} \right\} \quad (27)$$

Then the DMU_k is located on the VRS inefficient frontier.

(ii) If there is a combination of $\tilde{\upsilon}_i \geq 0$, $\tilde{\mu}_r \geq 0$, and $\tilde{\upsilon}_o$ weights so that

$$\frac{\sum_{i=1}^{m}\tilde{\upsilon}_i x_{ik} - \tilde{\upsilon}_o}{\sum_{r=1}^{s}\tilde{\mu}_r y_{rk}} = \max_j \left\{ \frac{\sum_{i=1}^{m}\tilde{\upsilon}_i x_{ij} - \tilde{\upsilon}_o}{\sum_{r=1}^{s}\tilde{\mu}_r y_{rj}} \right\} \quad (28)$$

Then the DMU_k is located on the VRS inefficient frontier.

Proof: This theorem is proved parallel to the theorem 1 using the fractional programming models (11) and (16).

In Theorem 2, if we assume (i) $\tilde{\mu}_o = 1$, $\tilde{\mu}_r = 0$ ($r = 1,\ldots,s$), $\tilde{\upsilon}_i = 1$ for $i \in I$, and $\tilde{\upsilon}_i = 0$ for $i \notin I$; or (ii) $\tilde{\upsilon}_o = 1$, $\tilde{\upsilon}_i = 0$ ($i = 1,\ldots,m$), $\tilde{\mu}_r = 1$ for $r \in R$, and $\tilde{\mu}_r = 0$ for $r \notin R$, then we immediately have:

Result 4:

(i) DMU_k is located on the VRS inefficient frontier if

$$\sum_{i \in I} x_{ik} = \max_j \left\{ \sum_{i \in I} x_{ij} \right\} \quad (29)$$

where $I \subseteq \{1,\ldots,m\}$ is a subset of inputs.

(ii) DMU_k is located on the VRS inefficient frontier if

$$\sum_{r \in R} y_{rk} = \min_j \left\{ \sum_{r \in R} y_{rj} \right\} \quad (30)$$

where $R \subseteq \{1,\ldots,s\}$ is a subset of outputs.

In addition, if $I = \{1,\ldots,m\}$ and $R = \{1,\ldots,s\}$, then:

Results 5:

(i) DMU_k is located on the VRS inefficient frontier, if

$$\sum_{i=1}^{m} x_{ik} = \max_j \left\{ \sum_{i=1}^{m} x_{ij} \right\} \quad (31)$$

Located on the inertial border of the VRS

(ii) DMU_k is located on the VRS inefficient frontier, if

$$\sum_{r=1}^{s} y_{rk} = \min_j \left\{ \sum_{r=1}^{s} y_{rj} \right\} \quad (32)$$

If I and R only have one component, then the following results are obtained:

Result 6:
(i) DMU_k is located on the VRS inefficient frontier if $x_{ik} = \max_j\{x_{ij}\}$.
(ii) DMU_k is located on the VRS inefficient frontier if $y_{rk} = \min_j\{y_{rj}\}$.

4 An experimental example

To emphasize the practical application of this method, it was applied to a data set consisting of 11 Iranian gas companies (DMUs) in 11 areas of Iran. The data used in this analysis were derived from the operations in 2003 and 2004. Five variables from the data set were used as inputs and outputs. The inputs included the budget (x_1) and the number of staff (x_2) and the outputs included amount of piping (y_1), the number of new customers (y_2), and the amount of branch-line (y_3). The data set was derived from Amirteimoori [20] and shown in Table 1.

Table 1 Normalized data set of eleven DMUs.

DMU	Inputs		Outputs		
	Budget	Number of staff	Amount of piping	number of new customers	Amount of branch-line
1	0.8973	0.9698	1.0000	0.3077	0.474
2	0.3884	0.9943	0.5325	0.4978	0.3953
3	0.7864	1.0000	0.2555	0.2935	0.354
4	0.6879	0.7926	0.9130	1.0000	0.9919
5	1.0000	0.7082	0.9385	0.8206	0.5763
6	0.9662	0.6008	0.2656	0.3473	0.2137
7	0.8261	0.6131	0.5658	0.5917	0.5922
8	0.9169	0.9416	0.4614	0.4863	0.4912
9	0.6223	0.4477	0.3408	0.6628	0.3208
10	0.8813	0.7639	0.8819	0.979	1.0000
11	0.8876	0.9870	0.7945	0.6105	0.5994

The DEA model (6) was applied to each of the eleven DMUs in Table 1 to achieve their pessimistic efficiency or the worst possible relative efficiency shown in the second column of Table 2. Since the pessimistic efficiencies for DMU_1, DMU_3, and DMU_6 equal 1, these DMUs are identified as pessimistic inefficient or DEA-inefficient with the worst performance among eleven DMUs. The remaining eight DMUs are called pessimistic non-inefficient or DEA-non-inefficient. The performance of these eight non-inefficient units is better than three inefficient units.

Table 2 Evaluation of eleven DMUs using ratio analysis.

DMU	Pessimistic efficiency (model (6))	Output–input ratio			
		y_1/x_1	y_2/x_2	y_2/x_1	$y_3/(x_1+x_2)$
1	1.0000	1.1145	0.3173	0.3429*	0.2539
2	1.1231	1.3710	0.5007	1.2817	0.2859
3	1.0000	0.3249	0.2935*	0.3732	0.1982
4	3.5335	1.3272	1.2617	1.4537	0.6700
5	2.2735	0.9385	1.1587	0.8206	0.3374
6	1.0000	0.2749*	0.5781	0.3594	0.1364*
7	1.9894	0.6849	0.9651	0.7163	0.4115
8	1.4458	0.5032	0.5165	0.5304	0.2643
9	1.9392	0.5476	1.4805	1.0651	0.2998
10	3.0557	1.0007	1.2816	1.1109	0.6078
11	1.7144	0.8951	0.6185	0.6878	0.3197

* indicates the minimum ratio which in turn indicates that the DMU is located on the CRS inefficient frontier.

To identify all DMUs located on the CRS inefficient frontier based on output/input ratio, the last four columns respectively represent y_1/x_1, y_2/x_2, y_2/x_1, and $y_3/(x_1+x_2)$. The first three ratios are based on the Result 3. The last ratio is based on the Result 1 where $\tilde{\mu}_1=0$, $\tilde{\mu}_2=0$, $\tilde{\mu}_3=1$, and $\tilde{\upsilon}_1=\tilde{\upsilon}_2=1$. The minimum values of these ratios indicate that the DMUs 1, 3, and 6 are located on the CRS inefficient frontier. According to Ali [8], identification of frontier DMUs before DEA calculation has a great impact on simplifying DEA solution. Therefore, the residual DEA scores are calculated more effectively and efficiently.

5 Conclusion

The present study showed the relationship between the ratio analysis and the pessimistic efficiency. DEA consists of a ratio analysis condition, that is, the DMUs with the lowest rank in terms of single output-single input ratio are dominated by the other DMUs. Such DMUs can easily be identified as a subset of frontier-forming DMUs in DEA. This study also showed the intrinsic defect of the ratio analysis. That is, it cannot identify all types of dominated DMUs like the DEA. As a result, a performance measure based on the single output-single input ratio is not able to express the overall performance compared to a set of outputs and inputs.

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