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Calculating the efficiency and productivity of decision making units with negative data

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Abstract Although non-negative data are fundamentally indispensable for determining the Malmquist productivity index (MPI), the observed values are sometimes negative in the real-world problems. In this paper, we reformulate the conventional Malmquist productivity index in data envelopment analysis (DEA) problem with negative data. So, first we want to introduce a non-radial efficiency model with negative data, then we use it in the Malmquist productivity index. At the end, we have tested the new proposed approach by the case study and applied to the productivity analysis of the 28 cement companies where are located in Iran's Burs evolved between 2012 and 2013, because some of these companies have one negative output. In the analysis of the case study, we show that the index Malmquist (productivity) of companies is measured correctly. And because the index Malmquist calculations are done by computing efficiency, therefore it can be resulted that the efficiency of companies with negative data is measured correctly.

Keyword: Data Envelopment Analysis (DEA), Decision Making Units (DMUs), Malmquist Productivity index (MPI), Negative data, Efficiency.

1 Introduction

Data envelopment analysis (DEA) is an approach for measuring the relative efficiency of group of decision making units (DMUs) with multiple inputs and multiple outputs using mathematical programming. Charnes et al. [1] originally proposed the first DEA model; this model had been known as the CCR model. Since then, a number of DEA models have been developed and a significantly large number of applications have been reported in the DEA literature. Conventional DEA models suppose non-negative values for inputs and outputs. However, there are many applications in which one or more inputs or outputs are necessarily negative such as loss when net profit is an output variable. Many real-world applications of DEA could be found in which we faced output variables, including both positive and negative values.

In DEA literature, there have been various approaches about dealing with negative data. Emrouznejad [2] has suggested Semi-Oriented Radial Measure (SORM) to handle variables that took both positive and negative values over the units. This model has given each input/output variable basically as a sum of two variables, one of them took negative and

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another one took positive values. Then Emrouznejad et al. [3] have investigated the necessary and sufficient conditions for the roundness of the input and output orientations of the variable returns to scale SORM DEA model. Kazemi Matin et al. [4, 5] have shown that the standard SORM has some inherent limitation. Afterward, they have introduced a new model for inefficiency evaluation and target setting, but their model didn't gain the efficiency score, therefore Jahanshahloo et al. [6] have modified their model for giving the efficiency score with negative data. This model was just input-oriented or output-oriented.

Kaffash et al. [7] modified the idea under a directional distance function framework. Similar to the case in Halme et al. [8], where it was found that increasing the number of factors may increase the efficiency score, it was noted that this method may not necessarily identify all efficient targets for inefficient DMUs to make improvements. Lin et al. [9] proposed an improved Super SBM model and the corresponding improved SBM model under the condition of variable returns to scale, both of which were feasible and allow input-output variables to take negative values. Their proposed approach has some advantages over the presence of negative data and successfully overcomes the drawbacks of the current super-efficiency models capable of handling negative data and extends Super SBM to the situation where negative data exist.

The idea of Portela et al. [10] was applied by Diabat et al. [11] to measure SBM efficiency, with the exception of the distance parameter was allowed to be different for each factor and by Tavana et al. [12] to propose a new directional measure of dynamic range (RDM) for two-stages of DEA models that allowed negative data as well as both desirable and undesirable carryover. Lin and Liu [13] provided the conditions to be satisfied by directions, with which the super-efficiency model was feasible and yields bounded super-efficiency scores, no matter there is negative data or there is not. Based on these, two types of directions were constructed. The (Directional distance function) DDF-based super-efficiency models with these restricted-function, super-efficiency scores for all the DMUs are capable of dealing with negative data well.

Kao [14] proposed a generalized radial model to define a more possibility of general production and set the only urgent need of aggregate input and aggregate output to be positive. The model can be used to identify unrealistic production processes. It works under the assumptions of both constant and variable returns to scale. It can be used to measure scale efficiency in addition to the conventional productive efficiency. This model can also be extended to network systems.

This method has a limitation of being applicable only in cases of the aggregate input and aggregate output are positive. The method fails when there are some peculiar DMUs which violate these conditions.

The computation of productivity has been changed into efficiency measures for the first time by Caves et al. [15] and developed by Nishimizu and Page [16] and by Färe et al. [17], in the context of parametric and non-parametric efficiency measurement, respectively. The Färe et al. [17] approach has become known as the measurement of changing of productivity through Malmquist indices. Though several applications of Malmquist indices exist in the literature, for the authors' knowledge, there was nowhere for efficiency measures to be computed for some situations where some data were negative. However, in real situations, data can be negative and therefore it is interesting that tools of efficiency measurement and productivity change analysis are developed to deal with such data. Until 2010, Portela and Thanassoulis [18] were the only individuals to develop an index of productivity change that can be used with negative data.

To measure efficiency under negative data, they used the approach that had been developed by Portela et al. [10] named range directional model (RDM). To calculate Malmquist indices using the RDM, they adapted the Global Malmquist index of Pastor and Lovell [19], analyzed and extended in Portela and Thanassoulis [20]. The index used a frontier of a single reference on a pooled panel of data. These results in a circular index of productivity change have been shown in Berg et al. [21]. They referred to our productivity index as meta-Malmquist index since the frontier of a pooled panel is often referred to as a meta-frontier.

While all the existing methods for handling negative data have merit, they also have drawbacks and limitations, the fact is, particularly that the economic foundations are weak, nevertheless, they are mathematically correct. In this current essay, we are going to propose the efficiency model like an enhanced Russell efficiency measures with the negative inputs and outputs that doesn't have their limitation. Afterward, we used this model for measuring a non-radial MPI, then we tested the new proposed approach a numerical example and applied to the productivity analysis of the 28 cement companies where are located in Iran's Burs evolved between 1391 and 1392, because some of these companies have one negative output.

The rest of the study is organized as follows: Section 2 briefly reviews the enhanced Russell efficiency model, some recent approaches about dealing with negative data in DEA and Non-Radial Malmquist productivity index. In the methodology section (Section 3), we developed the model of Jahanshahloo et al. [6] to enhanced Russell model and we reformulated Non-radial Mamquist productivity index with negative data. The proposed efficiency model and MPI are tested in the case study and applied to the productivity analysis of the 28 cement companies where are located in Iran's Burs evolved between 2012 and 2013 in Section 4. This paper concludes 5 Sections.

2 Background

2.1 Enhanced Russell Efficiency

Suppose we have *n* DMUs and each DMU_j $\left(j=1,...,n \right)$ uses a column vector of inputs $\left(X_j \right)$in order to yield a column of outputs$\left(Y_j \right)$, where $X_j = \left(x_{1j}, x_{2j}, \ldots, x_{mj} \right)^T$ and $Y_j = \left(y_{1j}, y_{2j}, \ldots, y_{sj} \right)^T$. It is also assumed that $X_j \geq 0,\ Y_j \geq 0,\ X_j \neq 0$ and $Y_j \neq 0$ for every $j = 1, \ldots, n.$ The following "enhanced Russell graph measure" model is a non-radial model under variable return to scale that Pastor et.al [22] introduced to measure the *DEA* technical efficiency of the *o*th $DMU\left(X_O, Y_O \right)$ $\left(O \in \{1, 2, \ldots, n\} \right)$.

$$
\begin{aligned}
&Min && R = \frac{\sum_{i=1}^{m} \theta_i \big/ m}{\sum_{r=1}^{s} \varphi_r \big/ s} && \\
&s.t && \sum_{j=1}^{n} x_{ij} \lambda_j \leq \theta_i x_{io}, && i = 1, 2, \cdots, m, \\
& && \sum_{j=1}^{n} y_{rj} \lambda_j \leq \varphi_r y_{ro}, && r = 1, 2, \cdots, s,
\end{aligned}
\qquad (1)
$$

$$\sum_{j=1}^{n}\lambda_j = 1,$$
$$\theta_i \le 1, \qquad i = 1, 2, \cdots, m,$$
$$\varphi_r \ge 1, \qquad r = 1, 2, \cdots, s,$$
$$\lambda_j \ge 0, \qquad j = 1, 2, \cdots, n.$$

Model (1) is a fractional programming structure. Charnes and Cooper [23] transformed the nonlinear model in (1) into an ordinary linear programming formulation as follows:

$$
\begin{aligned}
E_o = Min \quad & \sum_{i=1}^{m} \Theta_i / m \\
s.t \quad & \sum_{r=1}^{s} \Phi_r = s, \\
& \sum_{j=1}^{n} x_{ij}\Lambda_j \le \Theta_i x_{io}, && i = 1, 2, \cdots, m, \\
& \sum_{j=1}^{n} y_{rj}\Lambda_j \ge \Phi_r y_{ro}, && r = 1, 2, \cdots, s, \\
& \Theta_i \le \beta, && i = 1, 2, \cdots, m, \\
& \beta \le \Phi_r, && r = 1, 2, \cdots, s, \\
& \Lambda_j \ge 0, && j = 1, 2, \cdots, n, \\
& 0 \le \beta \le 1.
\end{aligned}
\tag{2}
$$

2.2 Some recent approaches to deal with negative data in DEA

We assume that the production process yields a portion containing both positive and negative data. This could be occurred in both input and output. That is, we have an input (output) that takes positive values for some and negative values for other DMUs. So let us to partition the observed input vector $\mathbf{x}_j$ as $\left(\mathbf{x}_j^P, \mathbf{x}_j^N\right)(j = 1, \ldots, n)$ and the observed output vector $\mathbf{y}_j$ as $\left(\mathbf{y}_j^P, \mathbf{y}_j^N\right)(j = 1, \ldots, n)$ where P is associated with the positive inputs (outputs) and N is related to the negative inputs (outputs).

2.2.1 A Semi- Oriented measure (SORM) to deal with negative data

Emrouznejad et al. [2] replaced $\mathbf{y}_j^N$ by $\mathbf{y}_j^N = \left(\mathbf{y}_j^1, \mathbf{y}_j^2\right)$ for $(j = 1, \ldots, n)$, where,

$$\mathrm{y}^1_{\mathrm{rj}}=\begin{cases} y_{rj} & \text{if} \quad y_{rj}\ge 0\\ 0 & \text{otherwise}\end{cases} \text{ and } \mathrm{y}^2_{\mathrm{rj}}=\begin{cases} -y_{rj} & \text{if} \quad y_{rj}< 0\\ 0 & \text{otherwise}\end{cases}.$$

Similarly they replaced $\mathbf{x}^{\mathrm{N}}_{\mathrm{j}}$ by $\mathbf{x}^{\mathrm{N}}_{\mathrm{j}}=\left(\mathbf{x}^1_{\mathrm{j}},\mathbf{x}^2_{\mathrm{j}}\right)$ for $\left(j=1,\ldots,n\right)$, where,

$$\mathrm{x}^1_{\mathrm{ij}}=\begin{cases} x_{ij} & \text{if} \quad x_{ij}\ge 0\\ 0 & \text{otherwise}\end{cases} \text{ and } \mathrm{x}^2_{\mathrm{ij}}=\begin{cases} -x_{ij} & \text{if} \quad x_{ij}< 0\\ 0 & \text{otherwise}\end{cases}.$$

Input oriented VRS SORM for evaluation $\mathrm{DMU}_k\left(k\in J=\{1,\ldots,n\}\right)$, when DMUs have positive and negative input and output variables is as follows:

$$\begin{aligned} Min \quad & h \\ s.t \quad & \sum_{j=1}^{n}\mathbf{x}^{\mathrm{P}}_{\mathrm{j}}\lambda_j \le h\mathbf{x}^{\mathrm{P}}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{x}^{1}_{\mathrm{j}}\lambda_j \le h\mathbf{x}^{1}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{x}^{2}_{\mathrm{j}}\lambda_j \ge h\mathbf{x}^{2}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{y}^{\mathrm{P}}_{\mathrm{j}}\lambda_j \ge \mathbf{y}^{\mathrm{P}}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{y}^{1}_{\mathrm{j}}\lambda_j \ge \mathbf{y}^{1}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{y}^{2}_{\mathrm{j}}\lambda_j \le \mathbf{y}^{2}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\lambda_j =1, \\ & \lambda_j \ge 0, \quad \forall j\in J. \end{aligned} \tag{3}$$

Output oriented VRS SORM for evaluating DMU_k $(k\in J)$, when DMUs have positive and negative input and output variables is as follows:

$$\begin{aligned} Max \quad & h \\ \text{s.t} \quad & \sum_{j=1}^{n}\mathbf{x}^{\mathrm{P}}_{\mathrm{j}}\lambda_j \le \mathbf{x}^{\mathrm{P}}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{x}^{1}_{\mathrm{j}}\lambda_j \le \mathbf{x}^{1}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{x}^{2}_{\mathrm{j}}\lambda_j \ge \mathbf{x}^{2}_{\mathrm{k}} \\ & \sum_{j=1}^{n}\mathbf{y}^{\mathrm{P}}_{\mathrm{j}}\lambda_j \ge h\mathbf{y}^{\mathrm{P}}_{\mathrm{k}} \end{aligned} \tag{4}$$

$$\sum_{j=1}^{n} \mathbf{y}_j^1 \lambda_j \geq h\mathbf{y}_k^1$$
$$\sum_{j=1}^{n} \mathbf{y}_j^2 \lambda_j \leq h\mathbf{y}_k^2$$
$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\lambda_j \geq 0, \quad \forall j \in J.$$

2.2.2 A modified SORM model

Kazemi matin et al. [4, 5] highlighted the problem in efficiency evaluation and setting targets in the standard SORM model. They presented $y_{rk}^1 = 0$ and $y_{rk}^2 > 0$ for each negative output of DMU_K, if DMU_K is inefficient, then $h^* y_{rk}^2 > y_{rk}^2$ and the target set of DMU_K has a value of $h^*(y_{rk}^1 - y_{rk}^2) = h^* y_{rk}^2 < -y_{rk}^2 = y_{rk}$, therefore the target output is poorer, than the actual itself value. Therefore, they introduced two modified SORM models.
First model (Kazemi Matin et al. [4]):

$$\begin{aligned}
Max \quad & h \\
s.t \quad & \sum_{j=1}^{n} \mathbf{x}_j^P \lambda_j \leq (1-h)\mathbf{x}_k^P \\
& \sum_{j=1}^{n} \mathbf{x}_j^1 \lambda_j \leq (1-h)\mathbf{x}_k^1 \\
& \sum_{j=1}^{n} \mathbf{x}_j^2 \lambda_j \geq (1+h)\mathbf{x}_k^2 \\
& \sum_{j=1}^{n} \mathbf{y}_j^P \lambda_j \geq \mathbf{y}_k^P \\
& \sum_{j=1}^{n} \mathbf{y}_j^1 \lambda_j \geq \mathbf{y}_k^1 \\
& \sum_{j=1}^{n} \mathbf{y}_j^2 \lambda_j \leq \mathbf{y}_k^2 \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad \forall j \in J.
\end{aligned} \tag{5}$$

Second model (Kazemi Matin et al. [5]):

$$Max \quad h$$

$$
\begin{aligned}
s.t \quad & \sum_{j=1}^{n} \mathbf{x}_j^P \lambda_j \le \mathbf{x}_k^P \\
& \sum_{j=1}^{n} \mathbf{x}_j^1 \lambda_j \le \mathbf{x}_k^1 \\
& \sum_{j=1}^{n} \mathbf{x}_j^2 \lambda_j \ge \mathbf{x}_k^2 \\
& \sum_{j=1}^{n} \mathbf{y}_j^P \lambda_j \ge h\mathbf{y}_k^P \\
& \sum_{j=1}^{n} \mathbf{y}_j^1 \lambda_j \ge h\mathbf{y}_k^1 \\
& \sum_{j=1}^{n} \mathbf{y}_j^2 \lambda_j \le \frac{1}{h}\mathbf{y}_k^2 \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0, \quad \forall j \in J.
\end{aligned} \tag{6}
$$

But model (5) didn't calculate the efficiency score of DMUs and model (6) is a nonlinear problem.

2.2.3 Jahanshahloo et al.'s efficiency model with the negative data

Jahanshahloo et al. [6], has modified the model of Kazemi Matin [4] for obtaining DMUs' efficiency score. They set $h' = 1 - h$ in the model (4), and then proposed the following model: If DMU_K is an efficient unit in model (7), then $\theta'^* = 1$, other less $0 < \theta'^* < 1$.

$$
\begin{aligned}
Min \quad & \theta' \\
s.t \quad & \sum_{j=1}^{n} x_j^P \lambda_j \le \theta' x_k^P \\
& \sum_{j=1}^{n} x_j^1 \lambda_j \le \theta' x_k^1 \\
& \sum_{j=1}^{n} x_j^2 \lambda_j \ge (2 - \theta') x_k^2 \\
& \sum_{j=1}^{n} y_j^P \lambda_j \ge y_k^P \\
& \sum_{j=1}^{n} y_j^1 \lambda_j \ge y_k^1 \\
& \sum_{j=1}^{n} y_j^2 \lambda_j \le y_k^2 \\
& \sum_{j=1}^{n} \lambda_j = 1,
\end{aligned} \tag{7}
$$

$$\lambda_j \geq 0, \quad \forall j \in J.$$

2.3 Non-Radial Malmquist productivity index

Suppose we have n DMUs, each $DMU_K \left(K \in J = \{1, 2, \ldots, n\}\right)$, producing a vector of outputs $Y_K^t = \left(y_{1K}^t y_{2K}^t, \ldots, y_{sK}^t\right)$ by using a vector of inputs $X_K^t = \left(x_{1K}^t x_{2K}^t, \ldots, x_{mK}^t\right)$ at each time $t; t \in \{1, \ldots, T\}$. Now, we use the model of Pastor [22].

$$\begin{aligned}
D_K^t\left(X_K^t, Y_K^t\right) = Min \quad & R = \frac{\sum_{i=1}^{m}\theta_i / m}{\sum_{r=1}^{s}\varphi_r / s} \\
s.t \quad & \sum_{j=1}^{n} x_{ij}^t \lambda_j \leq \theta_i x_{iK}^t, \quad i = 1, 2, \cdots, m, \\
& \sum_{j=1}^{n} y_{rj}^t \lambda_j \leq \varphi_r y_{rK}^t, \quad r = 1, 2, \cdots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \theta_i \leq 1, \quad i = 1, 2, \cdots, m, \\
& \varphi_r \geq 1, \quad r = 1, 2, \cdots, s, \\
& \lambda_j \geq 0, \quad j = 1, 2, \cdots, n.
\end{aligned} \tag{8}$$

Where x_{iK}^t is the ith input and y_{rK}^t is the rth output for $\mathrm{DMU_K}$ in the period of time t. If we use $t+1$ instead of t in the above model, we have $D_K^{t+1}\left(X_K^{t+1}, Y_K^{t+1}\right)$ as the technical efficiency score for $\mathrm{DMU_K}$ in the period of time $t+1$. The technical efficiency for the first mixed period $D_K^t\left(X_K^{t+1}, Y_K^{t+1}\right)$ for each $\mathrm{DMU_K}$ $\left(K \in J = \{1, 2, \ldots, n\}\right)$, is obtained by solving:

$$\begin{aligned}
D_K^t\left(X_K^{t+1}, Y_K^{t+1}\right) = Min \quad & R = \frac{\sum_{i=1}^{m}\theta_i / m}{\sum_{r=1}^{s}\varphi_r / s} \\
s.t \quad & \sum_{j=1}^{n} x_{ij}^t \lambda_j \leq \theta_i x_{iK}^{t+1}, \quad i = 1, 2, \cdots, m, \\
& \sum_{j=1}^{n} y_{rj}^t \lambda_j \leq \varphi_r y_{rK}^{t+1}, \quad r = 1, 2, \cdots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, 2, \cdots, n.
\end{aligned} \tag{9}$$

Similarly, the other mixed period measure, $D_K^{t+1}\left(X_K^t, Y_K^t\right)$, compares $\left(X_K^t, Y_K^t\right)$ with the empirical production frontier in the time period $t+1$.
For calculating $D_K^t\left(X_K^{t+1}, Y_K^{t+1}\right)$ and $D_K^{t+1}\left(X_K^t, Y_K^t\right)$, DMU_K may be out of the PPS being considered. Therefore, constraints $0<\theta_i \le 1,$ and $\varphi_r \ge 1,$ imply that DMU_K moves away from the frontier of PPS , so the model become infeasible. To avoid this, these constraints are removed from the model. In fact, when we remove these two constraints, we let DMU_K to arrive at the frontier of PPS from out with increasing inputs and decreasing outputs.

Fare et al. [17] decomposed their Malmquist productivity index into two components:

$$M_K = [\frac{D_K^t(X_K^{(t+1)}, Y_K^{(t+1)})}{D_K^t(X_K^t, Y_K^t)} \times \frac{D_K^{(t+1)}(X_K^{(t+1)}, Y_K^{(t+1)})}{D_K^{(t+1)}(X_K^t, Y_K^t)}]^{(1/2)}$$

$$= \frac{D_K^{(t+1)}(X_K^{(t+1)}, Y_K^{(t+1)})}{D_K^t(X_K^t, Y_K^t)} \times [\frac{D_K^t(X_K^{(t+1)}, Y_K^{(t+1)})}{D_K^{(t+1)}(X_K^{(t+1)}, Y_K^{(t+1)})} \times \frac{D_K^t(X_K^t, Y_K^t)}{D_K^{(t+1)}(X_K^t, Y_K^t)}]^{(1/2)}.$$

The first component $\text{TEC}_\text{K} = \dfrac{D_K^{t+1}\left(X_K^{t+1}, Y_K^{t+1}\right)}{D_K^t\left(X_K^t, Y_K^t\right)}$ measures the change in technical efficiency. The second component, $\text{FS}_\text{K} = \left[\dfrac{D_K^t\left(X_K^{t+1}, Y_K^{t+1}\right)}{D_K^{t+1}\left(X_K^{t+1}, Y_K^{t+1}\right)} \times \dfrac{D_K^t\left(X_K^t, Y_K^t\right)}{D_K^{t+1}\left(X_K^t, Y_K^t\right)}\right]^{\frac{1}{2}}$ measures the technology frontier shift between periods of time t and $t+1$. If FS_K greater than one indicates a positive shift or technical progress, and if FS_K less than one indicate a negative shift or technical regress, and if FS_K equal to one indicate no shift in technology frontier.

Thus, Caves et al. [15] and Fare et al. [17] defined that $\text{M}_\text{K} > 1$ to indicate productivity gain, $\text{M}_\text{K} < 1$ to indicate productivity loss, and $\text{M}_\text{K} = 1$ to mean no change in productivity from time period t to $t+1$.

3 Methodology

In the background section, we introduced the newest papers about measuring efficiency of DMUs with the negative data. But they have some limitation for example, all of them were radial and had just one-oriented (input or output). In this current paper, we want to propose the efficiency model like a Russell's enhanced efficiency measure with the negative inputs and outputs that doesn't have their limitations. Afterward, we use this model for measuring a non-radial MPI.

3.1 Developing model of Jahanshahloo et al. (2011) to Russell's enhanced model

Now we are supposing that some inputs and outputs for some DMUs are negative, so the input and output vectors can be represented as:

$$\mathbf{x}_j = \left(\mathbf{x}_j^P, \mathbf{x}_j^N\right), \qquad \mathbf{y}_j = \left(\mathbf{y}_j^P, \mathbf{y}_j^N\right), \qquad (j = 1, \ldots, n),$$

where P is associated with the positive inputs (outputs) and N is related to the negative inputs (outputs). Now, we are going to calculate a non-radial model for obtaining DMUs' efficiency with non-oriented when they have negative data.

As you can see, we introduced model (7) in the section 2. Model (7) is an input-oriented model, so, the output-oriented of this model is as follows:

$$\begin{aligned}
Max \quad & \phi' \\
s.t \quad & \sum_{j=1}^{n} x_j^P \lambda_j \le x_k^P \\
& \sum_{j=1}^{n} x_j^1 \lambda_j \le x_k^1 \\
& \sum_{j=1}^{n} x_j^2 \lambda_j \ge x_k^2 \\
& \sum_{j=1}^{n} y_j^P \lambda_j \ge \phi' y_k^P \\
& \sum_{j=1}^{n} y_j^1 \lambda_j \ge \phi' y_k^1 \\
& \sum_{j=1}^{n} y_j^2 \lambda_j \le \left(2 - \phi'\right) y_k^2 \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0, \quad \forall j \in J.
\end{aligned} \tag{10}$$

Models (7) and (10) are radial models and input-oriented, output-oriented models, respectively. For calculating a non-radial and non-oriented model, we can combine two models (7) and (10) with Russell's enhanced model (model (1)), and obtain Russell's enhanced model with negative data, therefore the following model will obtain:

$$\begin{aligned}
D_K\left(X_K, Y_K\right) = Min \quad & E_K = \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{s}\sum_{r=1}^{s}\phi_r} \\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^P \le \theta_i x_{iK}^P \qquad i \in I^P \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^1 \le \theta_i x_{iK}^1 \qquad i \in I^N \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^2 \ge \ 2 - \theta_i \ x_{iK}^2 \qquad i \in I^N
\end{aligned} \tag{11}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{P} \geq \phi_r y_{rK}^{P} \qquad r \in R^P$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{1} \geq \phi_r y_{rK}^{1} \qquad r \in R^N$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{2} \leq \; 2-\phi_r \; y_{rK}^{2} \qquad r \in R^N$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \qquad j \in J$$

$$0 \leq \theta_i \leq 1, \qquad i \in I$$

$$\phi_r \geq 1, \qquad r \in R$$

Model (11) is a fractional programming structure. We could transform this nonlinear model into an ordinary linear programming formulation using the method of Charnes and Cooper [3]. Thus, the transformed model is as follows:

$$\mathrm{D_K}\left(X_K, Y_K\right) = Min \quad E_K = \frac{1}{m}\sum_{i=1}^{m} \Theta_i$$

$$s.t$$

$$\sum_{r=1}^{s} \Phi_r = s,$$

$$\sum_{j=1}^{n} \Lambda_j x_{ij}^{P} \leq \Theta_i x_{iK}^{P}, \qquad i \in I^P$$

$$\sum_{j=1}^{n} \Lambda_j x_{ij}^{1} \leq \Theta_i x_{iK}^{1}, \qquad i \in I^N$$

$$\sum_{j=1}^{n} \Lambda_j x_{ij}^{2} \geq \; 2l-\Theta_i \; x_{iK}^{2}, \qquad i \in I^N \tag{12}$$

$$\sum_{j=1}^{n} \Lambda_j y_{rj}^{P} \geq \Phi_r y_{rK}^{P}, \qquad r \in R^P$$

$$\sum_{j=1}^{n} \Lambda_j y_{rj}^{1} \geq \Phi_r y_{rK}^{1}, \qquad r \in R^N$$

$$\sum_{j=1}^{n} \Lambda_j y_{rj}^{2} \leq \; 2l-\Phi_r \; y_{rK}^{2}, \qquad r \in R^N$$

$$\sum_{j=1}^{n} \Lambda_j = l,$$

$$\Lambda_j \geq 0, \qquad j \in J$$

$$0 \leq \Theta_i \leq l, \qquad i \in I$$

$$\Phi_r \geq l > 0, \qquad r \in R$$

3.2 Non-radial Mamquist productivity index with negative data

In this section, we are going to develop the non-radial Malmquist index for DMUs with the negative data. Therefore, we calculate the non-radial Malmquist by the model (12) as we illustrated in subsection 2-2.

Suppose we have n DMUs, each DMU_K $\left(K \in J = \{1, 2, \ldots, n\}\right)$, producing a vector of outputs $Y_K^t = \left(\mathbf{y}_K^{Pt}, \mathbf{y}_K^{Nt}\right)$ by using a vector of inputs $X_K^t = \left(\mathbf{x}_K^{Pt}, \mathbf{x}_K^{Nt}\right)$ at each time $t; t \in \{1, \ldots, T\}$. Where P is associated with the positive inputs (outputs) and N is related to the negative inputs (outputs).

Also we replace $\mathbf{y}_K^{Nt}$ by $\mathbf{y}_K^{Nt} = \left(\mathbf{y}_K^{1t}, \mathbf{y}_K^{2t}\right)$ for $(j = 1, \ldots, n)$, where,

$$y_{rK}^{1t} = \begin{cases} y_{rK}^t & \text{if } \; y_{rk}^t \geq 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } y_{rK}^{2} = \begin{cases} -y_{rK}^t & \text{if } \; y_{rk}^t < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, we replace $\mathbf{x}_K^{Nt}$ with $\mathbf{x}_K^{Nt} = \left(\mathbf{x}_K^{1t}, \mathbf{x}_K^{2t}\right)$ for $(j = 1, \ldots, n)$, where,

$$x_{iK}^{1t} = \begin{cases} x_{iK}^t & \text{if } \; x_{iK}^t \geq 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } x_{iK}^{2t} = \begin{cases} -x_{iK}^t & \text{if } \; x_{iK}^t < 0 \\ 0 & \text{otherwise} \end{cases}.$$

So, $D_K^t\left(\mathbf{x}_K^{Pt}, \mathbf{x}_K^{Nt}, \mathbf{y}_K^{Pt}, \mathbf{y}_K^{Nt}\right)$ is obtained by the following model:

$$\begin{aligned}
D_K^t\left(\mathbf{x}_K^{Pt}, \mathbf{x}_K^{Nt}, \mathbf{y}_K^{Pt}, \mathbf{y}_K^{Nt}\right) = Min \quad & E_K = \frac{1}{m}\sum_{i=1}^{m}\Theta_i \\
s.t \quad & \sum_{r=1}^{s}\Phi_r = s, \\
& \sum_{j=1}^{n}\Lambda_j x_{ij}^{Pt} \leq \Theta_i x_{iK}^{Pt}, && i \in I^P \\
& \sum_{j=1}^{n}\Lambda_j x_{ij}^{1t} \leq \Theta_i x_{iK}^{1t}, && i \in I^N \\
& \sum_{j=1}^{n}\Lambda_j x_{ij}^{2t} \geq 2l - \Theta_i x_{iK}^{2t}, && i \in I^N \qquad (13) \\
& \sum_{j=1}^{n}\Lambda_j y_{rj}^{Pt} \geq \Phi_r y_{rK}^{Pt}, && r \in R^P \\
& \sum_{j=1}^{n}\Lambda_j y_{rj}^{1t} \geq \Phi_r y_{rK}^{1t}, && r \in R^N \\
& \sum_{j=1}^{n}\Lambda_j y_{rj}^{2t} \leq 2l - \Phi_r y_{rK}^{2t}, && r \in R^N \\
& \sum_{j=1}^{n}\Lambda_j = l, \\
& \Lambda_j \geq 0, && j \in J
\end{aligned}$$

$$0 \le \Theta_i \le l, \qquad i \in I$$
$$\Phi_r \ge l > 0, \qquad r \in R$$

It is clear that, if we use $t+1$ instead of t in the model (13), we will obtain $D_K^{t+1}\left(\mathbf{x}_K^{P(t+1)}, \mathbf{x}_K^{N(t+1)}, \mathbf{y}_K^{P(t+1)}, \mathbf{y}_K^{N(t+1)}\right)$.

The technical efficiency for the first and the second mixed period $(D_K^t\left(\mathbf{x}_K^{P(t+1)}, \mathbf{x}_K^{N(t+1)}, \mathbf{y}_K^{P(t+1)}, \mathbf{y}_K^{N(t+1)}\right)$, $D_K^{t+1}\left(\mathbf{x}_K^{Pt}, \mathbf{x}_K^{Nt}, \mathbf{y}_K^{Pt}, \mathbf{y}_K^{Nt}\right)$ for each DMU_K $\left(K \in J = \{1, 2, \ldots, n\}\right)$ are obtained as following models:

$$
\begin{aligned}
D_K^t\left(\mathbf{x}_K^{P(t+1)}, \mathbf{x}_K^{N(t+1)}, \mathbf{y}_K^{P(t+1)}, \mathbf{y}_K^{N(t+1)}\right) = Min \quad & E_K = \frac{1}{m}\sum_{i=1}^{m}\Theta_i & \\
s.t \quad & \sum_{r=1}^{s}\Phi_r = s, & \\
& \sum_{j=1}^{n}\Lambda_j x_{ij}^{Pt} \le \Theta_i x_{iK}^{P(t+1)}, & i \in I^P \\
& \sum_{j=1}^{n}\Lambda_j x_{ij}^{1t} \le \Theta_i x_{iK}^{1(t+1)}, & i \in I^N \\
& \sum_{j=1}^{n}\Lambda_j x_{ij}^{2t} \ge \left(2l - \Theta_i\right) x_{iK}^{2(t+1)}, & i \in I^N \qquad (14) \\
& \sum_{j=1}^{n}\Lambda_j y_{rj}^{Pt} \ge \Phi_r y_{rK}^{P(t+1)}, & r \in R^P \\
& \sum_{j=1}^{n}\Lambda_j y_{rj}^{1t} \ge \Phi_r y_{rK}^{1(t+1)}, & r \in R^N \\
& \sum_{j=1}^{n}\Lambda_j y_{rj}^{2t} \le \left(2l - \Phi_r\right) y_{rK}^{2(t+1)}, & r \in R^N \\
& \sum_{j=1}^{n}\Lambda_j = l, & \\
& \Lambda_j \ge 0, & j \in J
\end{aligned}
$$

and

$$
\begin{aligned}
D_K^{t+1}\left(\mathbf{x}_K^{Pt}, \mathbf{x}_K^{Nt}, \mathbf{y}_K^{Pt}, \mathbf{y}_K^{Nt}\right) = Min \quad & E_K = \frac{1}{m}\sum_{i=1}^{m}\Theta_i & \\
s.t \quad & \sum_{r=1}^{s}\Phi_r = s, & \\
& \sum_{j=1}^{n}\Lambda_j x_{ij}^{P(t+1)} \le \Theta_i x_{iK}^{Pt}, & i \in I^P
\end{aligned}
$$

$$\begin{aligned}
&\sum_{j=1}^{n}\Lambda_j x_{ij}^{1(t+1)} \le \Theta_i x_{iK}^{1t}, && i\in I^N \\
&\sum_{j=1}^{n}\Lambda_j x_{ij}^{2(t+1)} \ge\ 2l-\Theta_i\ x_{iK}^{2t}, && i\in I^N \\
&\sum_{j=1}^{n}\Lambda_j y_{rj}^{P(t+1)} \ge \Phi_r y_{rK}^{Pt}, && r\in R^P \\
&\sum_{j=1}^{n}\Lambda_j y_{rj}^{1(t+1)} \ge \Phi_r y_{rK}^{1t}, && r\in R^N \\
&\sum_{j=1}^{n}\Lambda_j y_{rj}^{2(t+1)} \le\ 2l-\Phi_r\ y_{rK}^{2t}, && r\in R^N \\
&\sum_{j=1}^{n}\Lambda_j = l, && \\
&\Lambda_j \ge 0, && j\in J
\end{aligned} \tag{15}$$

According to the definitions of the efficiency with negative data, the Malmquist productivity index changes as follows:

$$\mathrm{M_K}=\left[\frac{D_K^t\left(\mathbf{x}_K^{P(t+1)},\mathbf{x}_K^{N(t+1)},\mathbf{y}_K^{P(t+1)},\mathbf{y}_K^{N(t+1)}\right)}{D_K^t\left(\mathbf{x}_K^{Pt},\mathbf{x}_K^{Nt},\mathbf{y}_K^{Pt},\mathbf{y}_K^{Nt}\right)}\times\frac{D_K^{t+1}\left(\mathbf{x}_K^{P(t+1)},\mathbf{x}_K^{N(t+1)},\mathbf{y}_K^{P(t+1)},\mathbf{y}_K^{N(t+1)}\right)}{D_K^{t+1}\left(\mathbf{x}_K^{Pt},\mathbf{x}_K^{Nt},\mathbf{y}_K^{Pt},\mathbf{y}_K^{Nt}\right)}\right]^{\frac{1}{2}} \tag{16}$$

$$=\frac{D_K^{t+1}\left(\mathbf{x}_K^{P(t+1)},\mathbf{x}_K^{N(t+1)},\mathbf{y}_K^{P(t+1)},\mathbf{y}_K^{N(t+1)}\right)}{D_K^t\left(\mathbf{x}_K^{Pt},\mathbf{x}_K^{Nt},\mathbf{y}_K^{Pt},\mathbf{y}_K^{Nt}\right)}\times\left[\frac{D_K^t\left(\mathbf{x}_K^{P(t+1)},\mathbf{x}_K^{N(t+1)},\mathbf{y}_K^{P(t+1)},\mathbf{y}_K^{N(t+1)}\right)}{D_K^{t+1}\left(\mathbf{x}_K^{P(t+1)},\mathbf{x}_K^{N(t+1)},\mathbf{y}_K^{P(t+1)},\mathbf{y}_K^{N(t+1)}\right)}\times\frac{D_K^t\left(\mathbf{x}_K^{Pt},\mathbf{x}_K^{Nt},\mathbf{y}_K^{Pt},\mathbf{y}_K^{Nt}\right)}{D_K^{t+1}\left(\mathbf{x}_K^{Pt},\mathbf{x}_K^{Nt},\mathbf{y}_K^{Pt},\mathbf{y}_K^{Nt}\right)}\right]^{\frac{1}{2}}.$$

Due to previous definitions of the Malmquist index, $\mathrm{M_K}>1$ indicates productivity gain or technical progress, $\mathrm{M_K}<1$ indicates a productivity loss or technical regress, and $\mathrm{M_K}=1$ means no change in productivity from time period t to $t+1$.

4 The case study

In this day and age, the stock markets and the capital market are two of the most important economic growth factors of any country. The stock market is the showcase of the best industries and companies. The growth and slowdown of the stock exchange companies reflect the country’s economic fluctuations. One of the active companies in the stock exchange is the cement industry that it has suffered from a severe recession in the 90’s decade, so the cement companies have suffered from it a lot. Therefore the net profit of some companies was negative.

In this article, we are going to calculate the efficiency of 28 cement companies active in stock exchange in 2012 and 2013 by our proposed method, then we will compute their progress and regress of them over these two years. For evaluating the efficiency and productivity of these companies, we will consider three inputs (Expected Cost, Current Debt, Financial Costs) and three outputs (Sales, Net Profit, Current Assets) with our Malmquist method. The amount of inputs and outputs of these companies are given in tables 1 and 2.

As you have seen in table 1 and 2, the second outputs (Net profit) of three cement companies (Bagheren, Khoramabad, and Majd Khaef) were negative in 2012 and 2013, So we calculate the efficiency of these 28 cement companies by the model (12) for t=2012 and 2013, and place the obtained results in the third and fourth columns of Table 3, respectively. To calculate the Malmquist index, we need to estimate $D_K^t\left(\mathbf{x}_K^{P(t+1)},\mathbf{x}_K^{N(t+1)},\mathbf{y}_K^{P(t+1)},\mathbf{y}_K^{N(t+1)}\right)$ and $D_K^{t+1}\left(\mathbf{x}_K^{Pt},\mathbf{x}_K^{Nt},\mathbf{y}_K^{Pt},\mathbf{y}_K^{Nt}\right)$. Their values are calculated for 28 cement companies with negative data using models (13), (14) and are located in the fifth and sixth columns of Table 3. Consequently, we obtain the Malmquist index by equation (16) and put their result in the seventh column of Table 3. Finally, the progress and regress of companies are determined and set in the last column of Table 3.

Table1 Data of cement companies for the year 2012 (numbers are in millions of rails)

DMU	Name of the cement companies	Inputs			Outputs		
		I_1: Expected Cost	I_2: Current Debt	I_3: Financial Costs	O_1: Sales	O_2: Net Profit	O_3: Current Assets
1	Abiek	1,284,463	5,001,559	271122	2,301,750	223,772	997448
2	Orumieh	825,953	897,666	82657	1,296,590	322,504	554669
3	Esfahan	404,781	215,939	5578	608,349	188,906	416385
4	Bojnourd	651,987	821,030	94937	1,015,717	233,481	703320
5	Behbahan	499,534	216,152	3716	893,037	331,170	302821
6	Tehran	1,653,440	1,952,594	103815	2,442,793	855,690	1258956
7	Khash	451,416	284,570	13272	642,860	135,388	325192
8	Khazar	471,951	503,544	31955	687,576	113,678	278329
9	Khuzestan	1,062,901	1,900,102	127283	1,593,013	370,475	839279
10	Darab	552,291	395,681	17342	754,356	296,267	371843
11	Doroud	531,828	609,252	42320	724,156	99,705	407371
12	Shahroud	622,052	586,395	59270	1,048,349	291,176	537780
13	Shomal	668,495	703,994	43561	885,825	201,783	699540
14	Soufi	936,449	1,355,930	170669	1,441,685	234,805	459409
15	Gharb	592,940	582,361	19280	975,428	301,589	448923
16	Fars	365,703	320,718	11745	507,336	103,142	143006
17	Fars no	535,214	339,343	18902	814,826	216,443	316758
18	Ghaen	312,906	158,729	13655	513,301	223,341	280449
19	Karoun	315,765	498,331	2659	587,985	206,100	321350
20	Kerman	474,391	341,765	10766	754,789	290,475	448404
21	Mazandaran	1,160,281	980,728	76690	2,056,655	668,813	920115
22	Neyriz	145,683	78,492	3911	259,541	101,074	150878
23	Bagheran	0	859,467	127973	10	-146,231	122294
24	Khoramabad	0	68,563	0	0	-6,371	5380
25	Larestan	112,486	145,521	6742	113,921	6,358	100848
26	Majde khaef	96,800	149,384	1946	101,638	-7,998	43199
27	Momtazane Kerman	375,263	241,632	35145	601,360	130,688	405723
28	Gharbe Asia	1	318,734	0	1	2,880	42650

According to the results of Table 3, the efficiency of companies Bagheran and Khoramabad are equal one in 2012 and 2013. So, we can’t say anything about their progress and regression. In order to see that the results of our model are correct or not, we will, therefore, compare inputs and outputs 2012 to 2013 of these companies.

First, we compare the inputs and outputs of the Bagheran company. As you can see in Tables 1 and 2, the company’s first input is same two years, and equal to zero. The second input in the second year (2013) is a little more than the first year (2012), but the third input in the second year is too much less than the first year. To compare outputs, the first output in the second year is more than the first year, and unfortunately, the second output in the second year is more negative than the first year, but instead the third output is too much more than the output of the first year. By comparing, we understand that this company has more outputs by fewer inputs in the second year than in the first year. Therefore, this company has improved in the second year, and this conclusion is the same as the result of our model.

Table 2 Data of cement companies for the year 2013 (numbers are in millions of rails)

DMU	Name of the cement companies	Inputs			Outputs		
		I_1: Expected Cost	I_2: Current Debt	I_3: Financial Costs	O_1: Sales	O_2: Net Profit	O_3: Current Assets
1	Abiek	1,672,322	577,732	433447	2,683,691	452,443	1971571
2	Orumieh	1,052,050	760,526	72099	1,759,706	568,507	719247
3	Esfahan	496,884	280,373	3334	789,525	232,498	487757
4	Bojnourd	908,525	923,489	95943	1,475,240	378,731	990736
5	Behbahan	514,061	183,148	5417	1,086,622	510,205	421506
6	Tehran	2,106,014	2,607,008	139226	2,901,442	1,620,245	2101924
7	Khash	528,090	354,646	11665	827,498	240,794	496777
8	Khazar	672,629	366,898	26621	941,221	175,879	319615
9	Khuzestan	1,562,305	774,959	139062	2,373,906	597,224	1413250
10	Darab	625,843	407,718	17340	889,429	354,314	578604
11	Doroud	672,833	645,746	52931	972,748	145,968	526259
12	Shahroud	847,968	668,008	65169	1,486,996	481,946	777528
13	Shomal	708,821	883,482	32824	949,587	462,732	762782
14	Soufi	1,087,826	1,205,628	135973	1,721,924	335,674	536937
15	Gharb	676,973	370,682	42840	1,211,377	438,969	548417
16	Fars	532,374	247,057	14750	705,993	131,166	301187
17	Fars no	643,243	267,772	18755	1,115,190	429,114	469406
18	Ghaen	363,770	254,429	5156	566,976	237,491	358278
19	Karoun	459,349	599,355	218	827,877	297,121	414428
20	Kerman	601,810	396,355	10611	904,133	346,958	513317
21	Mazandaran	1,487,091	799,835	79736	2,837,297	1,044,701	1163055
22	Neyriz	186,689	87,858	3262	352,271	139,432	191109
23	Bagheran	0	1,050,109	3380	221	-22,157	118511
24	Khoramabad	0	92,187	0	0	-13,609	3115
25	Larestan	113,351	91,869	7055	179,603	49,669	129633
26	Majde khaef	47,450	200,320	2460	48,158	-16,188	138937
27	Momtazane Kerman	629,206	412,446	26004	941,223	198,848	965960
28	Gharbe Asia	1	279,100	0	1	2,224	36987

By Comparing the inputs and outputs of Khoramabad company in 2012 and 2013 by tables 1 and 2, we get that the first and the third inputs of this company are zero in both years, but the second input in the second year (2013) is more than the input of 2012. And by comparing the outputs, we realize that the first output is zero in 2013, while this output is positive in 2012, the second output in the second year is more negative than the second output in the first year (2012), the third output in 2013 is much less than the third output in 2012. Thus, this company has more outputs with less inputs in 2012 than in 2013, so, this information is going to show the decline of the company and this result is the same as the result of our model.

As you see in Table 3, Majd khaef cement company has the efficiency with value 0.413 and 1.000 in 2012 and 2013, respectively. You might think the company has progressed, but by comparing the inputs and outputs of this company in these two years, we can get that in

2013, the company relatively has fewer outputs with more inputs, so this company has not progressed. This conclusion is exactly the result of our model.

This example shows that the index Malmquist (productivity) of companies is measured correctly. And because the index Malmquist calculations are done by computing efficiency, therefore it can be resulted that the efficiency of companies is measured correctly.

Table 3 Efficiencies and Malmquest index of cement companies.

DMU	Name of the cement companies	Efficiency 2012	Efficiency 2013	$D_K^t(X_K^{t+1}, Y_K^{t+1})$	$D_K^{t+1}(X_K^t, Y_K^t)$	Malmquest index	results
1	Abiek	1.00000	1.00000	0.46586	0.12757	1.91093	progress
2	Orumieh	0.61496	0.62918	0.51623	0.26846	1.40266	progress
3	Esfahan	1.00000	1.00000	0.96921	0.68082	1.19314	progress
4	Bojnourd	1.00000	0.63718	0.41237	0.27280	0.98142	regress
5	Behbahan	1.00000	1.00000	1.33309	0.85569	1.23382	progress
6	Tehran	1.00000	1.00000	0.53726	0.30889	1.31884	progress
7	Khash	0.62485	0.69933	0.59764	0.41165	1.27471	progress
8	Khazar	0.43899	0.41605	0.37083	0.28014	1.12007	progress
9	Khuzestan	0.62466	1.00000	0.49064	0.22071	1.88647	progress
10	Darab	0.62430	0.68380	0.60553	0.42075	1.25551	progress
11	Doroud	0.47610	0.43168	0.27060	0.25819	0.94482	regress
12	Shahroud	0.73233	0.67482	0.54685	0.33900	1.21919	progress
13	Shomal	1.00000	0.67407	0.51170	0.27562	1.11867	progress
14	Soufi	0.47050	0.28241	0.29052	0.18012	0.98393	regress
15	Gharb	0.74847	0.64290	0.67078	0.39787	1.20339	progress
16	Fars	0.31607	0.49290	0.44325	0.35136	1.40261	progress
17	Fars no	0.56920	0.68769	0.79478	0.39098	1.56715	progress
18	Ghaen	1.00000	0.81185	0.82353	0.61807	1.04006	progress
19	Karoun	1.00000	1.00000	5.79778	0.70649	2.86469	progress
20	Kerman	1.00000	0.70798	0.65989	0.54493	0.92593	regress
21	Mazandaran	1.00000	1.00000	0.71982	0.37176	1.39150	progress
22	Neyriz	1.00000	1.00000	1.15864	0.93279	1.11451	progress
23	Bagheran	1.00000	1.00000	0.40939	0.11202	1.91169	progress
24	Khoramabad	1.00000	1.00000	0.61293	0.63742	0.98060	regress
25	Larestan	0.11349	1.00000	0.77935	0.52424	3.61920	progress
26	Majde khaef	0.41310	1.00000	0.14035	0.47020	0.85003	regress
27	Momtazane Kerman	1.00000	1.00000	0.45478	0.37779	1.09718	progress
28	Gharbe Asia	1.00000	1.00000	1.00872	1.09103	0.96154	regress

5 Conclusion

As a last result, comparing the relative performance of a set of DMUs at a specific period, conventional DEA can also be used to calculate the productivity change of a DMU with the Malmquist productivity index (MPI) model. While non-negative data are often used in conventional DEA, real-world data are sometimes negative. Consequently, there is a strong impetus for developing efficiency and productivity of DMUs with the negative data.

The standard DEA model cannot be used for efficiency of DMUs with negative data. In the background section, we have introduced some of the new efficiency models that can compute the efficiency of DMUs with negative data, but they have some limitations. We tried to propose the efficiency model with the negative data that doesn't have their limitations. Then we use it in the MPI. Something that could be found is reformulating the conventional Malmquist productivity index with negative data. At the end, we analyzed efficiency and the productivity growth of 28 cement companies where are located in Iran's Burs evolved between 2012 and 2013 by the proposed model, because some of these companies have one negative output. This example demonstrated that the Malmquist productivity index of companies is measured correctly. Because the productivity is measured by the MPI and defined as the ratio between efficiency for the same DMU in two different periods of time, therefore it can be shown that the efficiency of companies is measured correctly.

Few studies have been written about the calculation of the Malmquist productivity index with negative data. One of the famous papers is Portela and Thanassoulis [18]. They used the Rang Directional Model (RDM) with output-oriented to measure efficiency under negative data. Their model is radial model. Also, a constant return to scale (CRS) assumption for their technology isn't consistent with the existence of negative data. An important issue that must be taken into account is all of them didn't use the Malmquist productivity index, they used meta- Malmquist index.

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