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Numerical solution for the fifth-order KdV equation by using spectral methods

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Abstract Nonlinear wave equations are more difficult to study mathematically and that no general analytical method for their solution exists. It is found that the Exponential Time Differencing (ETD) scheme requires the least steps to achieve a given accuracy, offers a speedy method at calculation time, and has exceptional stability properties in solving a nonlinear equation. This article explains how we applied the exponential integrators (ETDRK4) to semi-linear problems to solve the fifth-order KdV equation. To solve, we define a new integrating factor e^{-ik^5t} and apply fast Fourier transform (FFT) for spatial discretization. For this purpose, we solve the diagonal example of the fifth-order KdV equation via the exponential time differencing Runge-Kutta 4 method (ETDRK4). Implementation of the method is proposed by short Matlab programs.

Keyword: Exponential Methods; Integration Factor Methods; Exponential Time Differencing Methods; Runge-Kutta Method.

1 Introduction

In all fields, whether science or engineering nonlinear phenomena, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on exist. It is observed that several non-linear evolution equations are generally used to describe these complex phenomena. Thus, the powerful and efficient methods to find numerical solutions and analytic solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists [1-7].

Cox and Matthews presented a clear derivation of the explicit ELP schemes of arbitrary order referring to the ELP methods as the Exponential Time Differencing (ETD) methods [8-10]. After that Tokman studied a class of exponential propagation techniques known as Exponential Propagation Iterative (EPI) schemes [11]. In order to make the ETD schemes better, Wright deliberated on these schemes and thus reformed the solution in integral form of a nonlinear autonomous system of ODEs to an extension in terms of matrix and vector functions products [12].

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The basic formula of the ETD schemes, the linear part of the differential equation is integrated and applied a suitable approximate for the nonlinear terms. Lawson [13] presented a similar approach for the first time that is currently used in the integrating factor (IF) schemes. In the approach of IF schemes [14-18], both sides of an ODE were multiplied by an integrating factor, and made a change of variable that the linear part can be solved exactly.

Applications of exponential time difference methods are in solving stiff systems. Moreover, comparing various fourth-order methods, including the ETD methods and their results revealed that the best choice was the ETD4RK method for solving various one-dimensional diffusion-type problems [19]. Aziz et al. [20-21] studied on the exponential time difference Runge-Kutta 4 method (ETDRK4) for solving the diagonal example of Korteweg-de Vries (KdV) and Kuramoto-Sivashinsky (K-S) equations [22-23] with Fourier's transformation,. Other papers on this subject include [24-32].

The present paper is organized as follows: In section 1, we present the subject. In section 2, we demonstrate background of the study in which related to a diagonal example.. In section 3, we carry out the implementation on a diagonal example of the fifth –order KdV equation, together with fast Fourier transform (FFT). In section 4, the exact and numerical solution is compared. In section 5, a brief conclusion is provided.

2. Background of the study

In this section, we intend to show a diagonal example, which is solved via spectral method [24]. The Burgers' equation is given

$$u_t - ju_{xx} + uu_x = 0 \qquad x \in [0,1]\,, \quad t \in [0,1] \tag{1}$$

with the initial and Dirichlet boundary conditions prescribed using

$$u(x,0) = (\sin(2\pi x))^2\,(1-x)^{\frac{3}{2}} \tag{2}$$

where $N = 500$, $j = 0.0003$ (for viscous Burgers' equation) and $j = 0$ (for inviscid Burgers' equation), $r = 0.03$ (the roots of unity in Matlab codes).

As a result of the periodic boundary condition, the problem can be reduced to a diagonal form by Fourier transformation.

In solving the problem, we can write

$$u_t - ju_{xx} + (\tfrac{1}{2}u^2)_x = 0. \tag{3}$$

In the above equation, we apply the Fast Fourier transform (FFT)

$$\hat{u}_t + jk^2\hat{u} + \tfrac{1}{2}ik\widehat{u^2} = 0 \tag{4}$$

Where $i = \sqrt{-1}$.

The equation (4) is multiplied by e^{jk^2t}, i.e.

$$e^{jk^2t}\,\hat{u}_t + e^{jk^2t}\varepsilon k^2\hat{u} + \tfrac{1}{2}ik\,e^{jk^2t}\,\widehat{u^2} = 0\ . \tag{5}$$

If we define the change of variable

$$\widehat{U} = e^{jk^2t}\hat{u} \tag{6}$$

with $\quad \widehat{U}_t = jk^2e^{jk^2t}\hat{u} + e^{jk^2t}\,\hat{u}_t \quad ,$ (7)

and substituting (7) in (5), we have

$$\widehat{U}_t + \tfrac{1}{2}ik\,e^{jk^2t}\,\widehat{u^2} = 0\ . \tag{8}$$

Working in Fourier space (applying FFT), the numerical algorithm discretizing can be obtained by

$$\widehat{U}_t + \tfrac{i}{2}e^{jk^2t}\,kF\big((F^{-1}\big(e^{-jk^2t}\widehat{U}\big))^2\big) = 0\ . \tag{9}$$

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where F is the Fourier transformed operator, and the following programme solves the diagonal of Burgers' equation. The Matlab program is proposed [24].

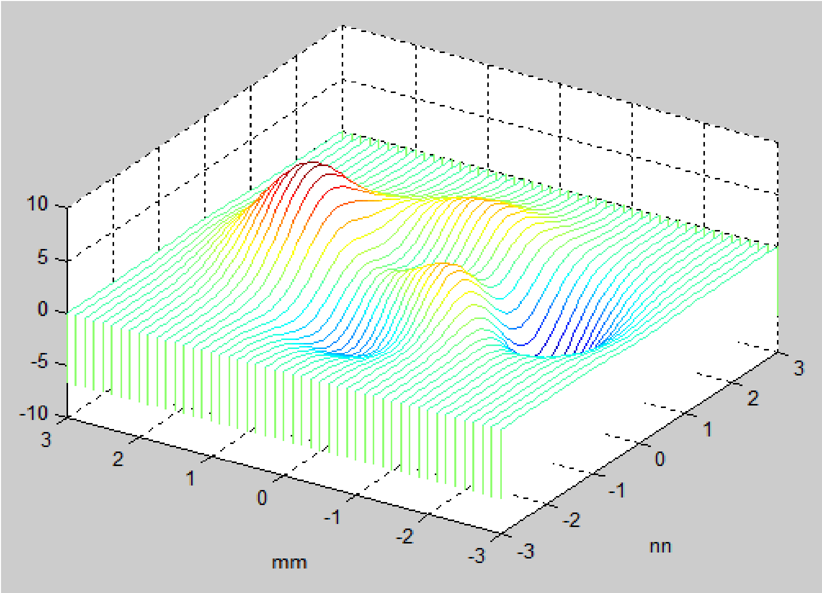

Fig.1 Time evolution for the inviscid Burgers equation ($j = 0$).The x axis runs from x = -3 to x = 3, and the t-axis runs from t = 0 to t = 150.

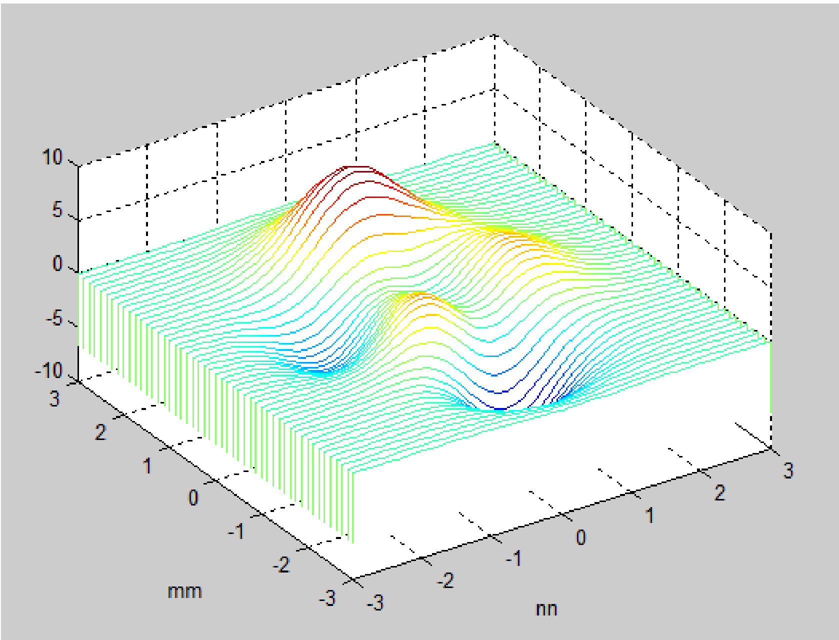

Fig. 2 Time evolution for the viscous Burgers' equation ($j \neq 0$), where the x axis runs from x = 3 to x = 3, and the t-axis runs from t = 0 to t = 150.

3. A diagonal example

The fifth-order kdv equation is considered as follows:

$$u_t + uu_x + u_{xxxxx} = 0 \qquad , x \in [-\pi, \pi] \tag{10}$$

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The equation (10) can be written as

$$u_t + (\frac{1}{2}u^2)_x + u_{xxxxx} = 0$$

With the initial condition given by

$$u(x,0) = -\frac{3}{2}A^2B^2sech^2\left(\frac{ABx}{2}\right) - \frac{17}{12}A^2B^2 \quad (11)$$

where A and are arbitrary constants.

In the above equation, the fast Fourier transform (FFT) is applied

$$\widehat{u_t} + \frac{ik}{2}\,\widehat{u^2} - ik^5\,\hat{u} = 0\ , \quad (12)$$

Where $\mathrm{i} = \sqrt{-1}$.

The equation (12) is multiplied by e^{-ik^5t} - the integrating factor-to find

$$e^{-ik^5t}\widehat{u_t} + e^{-ik^5t}\frac{ik}{2}\,\widehat{u^2} - e^{-ik^5t}ik^5\,\hat{u} = 0 \quad (13)$$

Let us consider

$$\widehat{U} = e^{-ik^5t}\,\hat{u},\ \text{With}\ \widehat{U_t} = -ik^5e^{-ik^5t}\hat{u} + e^{-ik^5t}\,\widehat{u_t}\,.$$

The equation (13) changes to the following form

$$\widehat{U_t} + ik^3\widehat{U} + \frac{i}{2}\,e^{-ik^5t}\,k\widehat{u^2} - ik^5\widehat{U} = 0 \quad (14)$$

i.e,

$$\widehat{U_t} + \frac{i}{2}\,e^{-ik^5t}\,k\widehat{u^2} = 0 \quad (15)$$

To work in Fourier's space, the numerical algorithm discretizing can be found by

$$\widehat{U_t} + \frac{i}{2}\,e^{-ik^5t}\,k\,F\left((F^{-1}(e^{-ik^5t}\widehat{U}\,))^2\right) = 0, \quad (16)$$

Where F is the Fourier transformed operator.

For time stepping, we use the ETDRK4 with $\mathrm{t} = 150$, the ETDRK4 is given by

$$\mathrm{a_n} = \mathrm{u_n}\mathrm{e}^{\mathrm{hL}/2} + \left(\mathrm{e}^{\mathrm{hL}/2} - \mathrm{I}\right)\mathrm{N}(\mathrm{u_n},\mathrm{t_n})/\mathrm{L}, \quad (17)$$

$$\mathrm{b_n} = \mathrm{u_n}\mathrm{e}^{\mathrm{hL}/2} + \left(\mathrm{e}^{\mathrm{hL}/2} - \mathrm{I}\right)\mathrm{N}(\mathrm{a_n},\mathrm{t_n} + \mathrm{h}/2\,)/\mathrm{L} \quad (18)$$

$$\mathrm{c_n} = \mathrm{a_n}\mathrm{e}^{\mathrm{hL}/2} + \left(\mathrm{e}^{\mathrm{hL}/2} - \mathrm{I}\right)(2\mathrm{N}(\mathrm{b_n},\mathrm{t_n} + \mathrm{h}/2\,) - \mathrm{N}(\mathrm{u_n},\mathrm{t_n}))\,/\mathrm{L} \quad (19)$$

$$\mathrm{u_{n+1}} = \mathrm{a_n}\mathrm{e}^{\mathrm{hL}} + \left\{ \begin{array}{l}\emptyset_1\mathrm{N}(\mathrm{u_n},\mathrm{t_n}) + 2\emptyset_2(\mathrm{N}(\mathrm{a_n},\mathrm{t_n} + \mathrm{h}/2\,) + \mathrm{N}(\mathrm{b_n},\mathrm{t_n} + \mathrm{h}/2\,)) \\ \qquad + \emptyset_3\,\mathrm{N}(\mathrm{c_n},\mathrm{t_n} + \mathrm{h})\end{array}\right\} \Big/ (\mathrm{L}^3\mathrm{h}^2), \quad (20)$$

Where

$$\emptyset_1 = (\mathrm{L}^2\mathrm{h}^2 - 3\mathrm{Lh} + 4)\mathrm{e}^{\mathrm{hL}} - \mathrm{Lh} - 4\,, \quad (21)$$

$$\emptyset_2 = (\mathrm{Lh} - 2)\mathrm{e}^{\mathrm{hL}} + \mathrm{Lh} + 2\,, \quad (22)$$

$$\emptyset_3 = (-\mathrm{Lh} + 4)\mathrm{e}^{\mathrm{hL}} - \mathrm{L}^2\mathrm{h}^2 - 3\mathrm{Lh} - 4 \quad (23)$$

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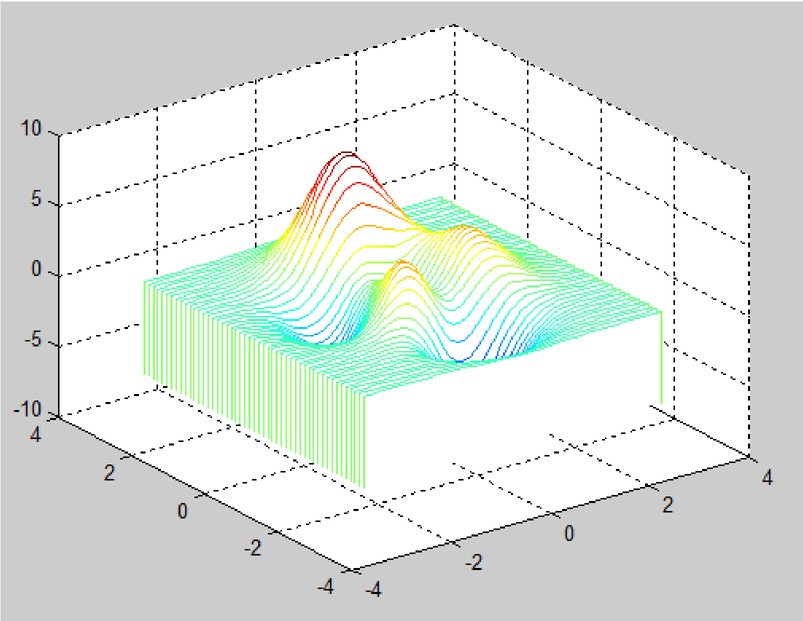

Fig. 3 Time Evolution for the fifth-order KdV equation. The X Axis Runs from $X = -\pi$ to $X = \pi$ and the t-Axis Runs from t=0 to t=150.

4. Comparison with the numerical solution and the exact solution

The fifth-order KdV equation (10) and initial condition (11) are presented. In Figure 4, the results of the equation (11) are plotted

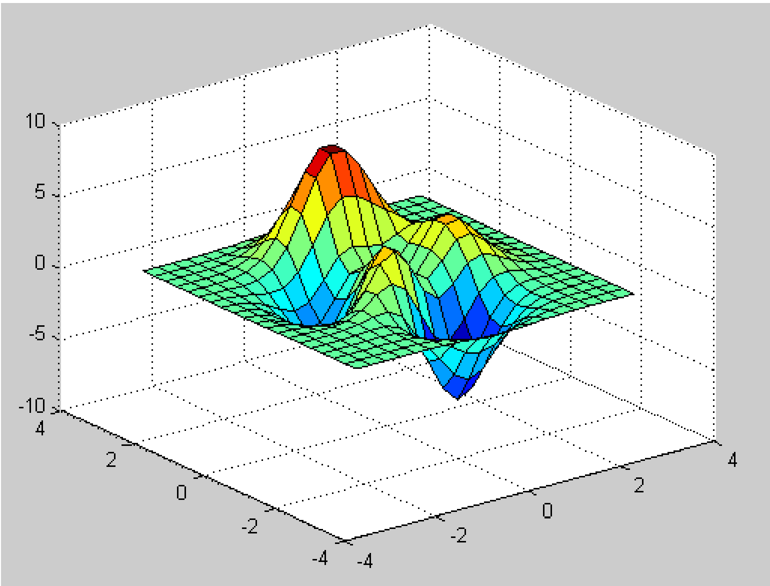

Fig. 4 The Evolution of Exact Solution the fifth-order KdV equation.

Figure 4 shows that the results of ETDRK4 numerical solution as shown in Figure 3 are suitable to the exact solution. In summary, the ETDRK4 is able to create the numerical solution.

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5. Conclusion

In this work, we have considered a nonlinear wave equation. We have presented the fifth-order KdV equation with the initial condition $u(x,0) = -\frac{3}{2}A^2B^2sech^2\left(\frac{ABx}{2}\right) - \frac{17}{12}A^2B^2$. To integrate the system (10), the ETDRK4 method is applied. To solve the equation, we illustrated the Matlab software. Figure 4 shows the result created by Matlab code listed (Appendix A). It is seen that ETDRK4 is suitable in every case. For diagonal and non-diagonal problems, it works equally well because it is accurate and fast, and the ETDRK4 needs the fewest steps to achieve a given accuracy. Furthermore, to run the programmes, the computational time required is less than one second, which is faster compared with the conventional Runge-Kutta 4.

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Appendix A.

Matlab code to solve the fifth-order kdv equation and produce Figure 1.

```
clear
clc
N = 256; dt = .4/N^2; x = (2*pi/N)*(-N/2:N/2-1);
A = 1; B = 1; clf, drawnow
u =- 3/2*A^2*B^2*sech(( A*B*x)/2).^2 - 17/12*B^2*A*2;
v= fft(u);
% precomputed various ETDRK4 scalar quantities:
k = [0:N/2-1 0 -N/2+1:-1];
ik5 = 1i*k.^5;
L=ik5;
% h = waitbar(0, 'please Wait. . . ' );
h=1/4;
E = exp (dt*ik5/2); E2 = E.^2;
M = 16; % no. of points for complex means
r = exp (1i*pi*((1:M)-.5)/M);
LR1= h*L (:,ones(M,1));
LR2= r (:, ones (M,1));
LR= LR1+LR2;
Q = h*real (mean( (exp(LR/2)-1)./LR ,2));
f1 = h*real(mean( (-4-LR+exp(LR).*(4-3*LR+LR.^2))./LR.^3 ,2));
f2 = h*real (mean( (2+LR+exp(LR).*(-2+LR))./LR.^3 ,2));
f3 = h*real (mean( (-4-3*LR-LR.^2+exp(LR).*(4-LR))./LR.^3 ,2));
% Main time-stepping loop:
uu = u; tt = 0;
g = -.5i*dt*k ;
%  Solve PDE and plot results:
tmax = 0.006 ; nplt = floor ( (tmax/25) /dt ) ; nmax = round(tmax/dt) ;
for n = 1:nmax
t = n*dt;
Nv = g.*fft(real(ifft(v)).^2);
a = E2.*v + Q.*Nv;
Na = g.*fft(real(ifft(a)).^2);
b = E2.*v + Q.*Na;
Nb = g.*fft(real(ifft(b)).^2);
c = E2.*a + Q.*(2*Nb-Nv);
Nc = g.*fft(real(ifft(c)).^2);
v = E.*v + Nv.*f1 + 2*(Na+Nb).*f2 + Nc.*f3;
if mod(n,nplt)==0
u = real(ifft(v));
uu = [uu,u]; tt = [tt ,t];
end
end
nn=length(tt);
mm=length(x);
uu2=reshape(uu,mm,nn);
figure
[mm,nn,uu2]=peaks;
waterfall (mm,nn,uu2);
colormap( [O 0 0] ) , view (-20,25);
xlabel x, ylabel t, axis([-pi pi 0 tmax 0 2000]), grid off
set (gca, 'ztick' , [O 2000] ) , close(h) , pbaspect ( [l 1 .13])
```


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