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Bernoulli Vacation Policy for a Bulk Retrial Queue with Fuzzy Parameters

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Abstract In this paper, we investigate the fuzzy logic based system characteristics of $M^X/G/1$ retrial queuing system with Bernoulli vacation schedule. The service time and vacation time are assumed to be generally distributed. It is found in many practical situations that the queuing models with fuzzy parameters are much more realistic than the classical crisp parameters based queuing models. We have chosen group arrival rate, service rate and vacation rate to be fuzzy parameters. The objective of this study in this paper is to transform $M^X/G/1$ retrial fuzzy queue with Bernoulli vacation schedule to a family of conventional crisp queues by employing α-cut approach based on Zadeh's extension principle. We formulate a pair of parametric non-linear programs (PNLPs) to describe the family of crisp queues with a vacationing server. To illustrate the validity of the proposed approach, the numerical examples are facilitated for different service time and vacation time distributions.

Keywords Retrial Queue, Batch Arrival, Bernoulli Vacation, Fuzzy Sets, Queue Length.

1 Introduction

Retrial queuing systems are characterized by the requirement of the customers who on finding the service area busy must join the retrial group and retry for service after a random interval of time. Queuing systems with retrial customers are quite common in many real world congestion situations, including web access, call centers, telephone switch systems, digital cellular mobile networks and computer networks, etc. Krishna Kumar and Arivudainambi [1] gave the analysis of a single server queue with Bernoulli vacation schedules and general retrial times. The detailed analysis of a retrial queuing model with optional phase type server vacations based on exhaustive deterministic service and a single vacation policy was done by Madan and Al-Rub [2]. Wenhui [3] considered the M/G/1 retrial queue with Bernoulli vacation and generic retrial, vacation, setup and service times. They have established the ergodic condition and obtained the probability generating function of the system size. Sherman and Kharoufeh [4] analyzed M/M/1 retrial queue with unreliable server whose normal and retrial queues have infinite storage capacity. An $M^X/G/1$ queuing system with two phases of heterogeneous service and Bernoulli vacation schedule which operate under classical retrial policy has been discussed by Choudhury [5]. This model generalizes both the classical M/G/1 retrial policy with arrivals in batches and a two phase batch arrival queue with single vacation under Bernoulli vacation schedule. Further in [6], he extended the same model to operate under the linear retrial policy. Recently, a batch arrival retrial queue with

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general retrial times, where the server is subject to starting failures and provides two phases of heterogeneous services to all customers under Bernoulli vacation schedules was investigated by Ke and Chang [7]. Moreover, Boualem et al. [8] have derived several stochastic comparison properties in the sense of strong stochastic ordering and convex ordering for an M/G/1 retrial queue with vacations.

While looking towards traditional queuing theory, we find that the inter-arrival times, service times and vacation times are required to follow certain probability distributions with fixed parameters. However, in many practical situations, the arrival pattern of the customers, the service pattern and vacation pattern of the server are typically described by linguistic values such as fast, slow or moderate rather than with complete probability distributions. Thus by considering some system parameters as fuzzy numbers, the queuing models so developed have wider range of applications in the real life. The literature of fuzzy queues is not much rich; although significant works have been done on fuzzy queues in the recent past. The works on fuzzy queues includes the analysis and design of membership functions for the system characteristics for some well known queuing models such as queues with batch arrival (cf. Chen [9]), retrial queues (cf. Ke et al.[10]), bulk service queues (cf. Chen [11]), unreliable server queues (cf. Ke and Lin [12]), queues with vacation (cf. Ke et al. [13]), finite-capacity queues (cf. Chen [14]) and their combinations and many others. It is worthwhile to mention some notable works on fuzzy queues in different frameworks. Li and Lee [15] proposed a general approach for queuing systems in a fuzzy environment based on Zadeh's extension principle. However, their approach is complicated and creates difficulty for applying in fuzzy queues with server vacation. Negi and Lee [16] proposed the α-cut and two-variable simulation approaches to solve some fuzzy queues; but their approaches have failed to describe the membership functions of various performance measures of these queues. Kao et al. [17] applied α-cut approach to reduce a fuzzy queue into a family of crisp queues.

Chen [11] was able to conserve the fuzziness of input information when some information of bulk service queuing systems is ambiguous. Further in [9], he developed a non-linear programming approach to derive the membership functions of the steady-state performance measures in bulk arrival queuing systems with varying batch sizes, wherein the arrival rate and service rate are fuzzy numbers. The membership functions of the system characteristics of a batch arrival queuing model with vacation policies were constructed by Ke et al. [13]. They suggested that by extending the classical queuing models with server vacation to the fuzzy environment, more information in the form of analytic results are provided which can then be easily implemented by the system designers and practitioners in real time situations. Further, in [10], Ke et al. constructed the membership functions of the system characteristics of a retrial queuing model with fuzzy arrival, retrial and service rates. They have facilitated a numerical example to validate the proposed approach. Lin and Ke [18] have developed parametric non-linear programs to convert a crisp controllable queuing model to the respective fuzzy model in which cost elements, arrival rate and service rate are all fuzzy numbers. Ritha and Robert [19] applied fuzzy set theory to estimate the uncertainty associated with the input parameters of M/M/1/1 retrial queue. They have used α-cut approach and fuzzy arithmetic operations to derive the system characteristics. Recently, Ritha and Menon [20] have proposed a procedure to construct a membership function of the performance measures of a queue with removable and reliable server. They have chosen arrival rate and service rate in the fuzzy environment.

In the present paper, we investigate the system characteristics of batch arrival retrial queuing system with Bernoulli vacation schedule by assuming the arrival, service and vacation rates as fuzzy numbers. The rest of the paper is organized as follows. In section 2,

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we formulate the model by stating the suitable assumptions. The fuzzy membership functions based on Zadeh's extension principle are described in section 3. Section 4 facilitates the numerical examples to explore the validity of analytical model to the real world congestion situations. Finally, in section 5, the paper is wind up with the conclusion and highlighting the scopes of the work done for future aspirants.

2 Problem Formulations

Consider an $M^X/G/1$queueing system wherein the single server follows Bernoulli vacation schedule. In this queuing system, the customers arrive in batches according to a compound Poisson process with group arrival rate $\tilde{\lambda}$, where $\tilde{\lambda}$ is a fuzzy number. The actual number of customers in any arriving batch is stochastically equivalent to a generic random variable X, where X may take positive values with probability f(X). Let c_1 and c_2 denote the first and second moments of the batch size. If the arriving batch finds the server free, then one of the customers from that batch is taken for service with rate $\tilde{\mu}$, where $\tilde{\mu}$ is a fuzzy number and the rest will join a pool of blocked customers called 'orbit' wherein the customers wait for random amount of time and then retry for service with rate θ. On the other hand, in case when the server is busy at the arrival epoch, then the whole batch joins the 'orbit'. After each service completion, the server has a choice either to go for vacation with probability p and with rate $\tilde{\nu}$ ($\tilde{\nu}$ is a fuzzy number) or continues to provide service to the next customer in the queue with probability $\bar{p} = 1 - p$. The inter-arrival time, service time and vacation time are assumed to be general distributed in the fuzzy environment.

3 The Solution Procedure

In this study, we construct the membership functions for the expected number of customers in the orbit and the expected waiting time of the customers for a batch arrival retrial queue with Bernoulli vacation schedule for different service and vacation time distributions. Other fuzzy performance measures can also be derived by using the proposed solution procedure.

3.1 Extension principle: Some Definitions

In the queuing model, we assume that group arrival rate ($\tilde{\lambda}$), service rate ($\tilde{\mu}$) and vacation rate ($\tilde{\nu}$) are fuzzy numbers which are approximately known. Then, we represent these fuzzy numbers as

$$\tilde{\lambda} = \left\{\left(x, \mu_{\tilde{\lambda}}(x)\right) / \, x \in S(\tilde{\lambda})\right\} \quad (1)$$

$$\tilde{\mu} = \left\{\left(y, \mu_{\tilde{\mu}}(y)\right) / \, y \in S(\tilde{\mu})\right\} \quad (2)$$

$$\tilde{\nu} = \left\{\left(w, \mu_{\tilde{\nu}}(w)\right) / \, w \in S(\tilde{\nu})\right\} \quad (3)$$

Here, $\mu_a(b)$ and S(a) denote the membership function and support of a where a= $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\nu}$ are the fuzzy numbers and b=x, y, w are the crisp values corresponding to group arrival rate, service rate and vacation rate, respectively.

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Let $P(x, y, w)$ and $\widetilde{P}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$ denote the system performance measures of interest in the crisp and fuzzy environments, respectively. As $\widetilde{\lambda}$, $\widetilde{\mu}$, $\widetilde{\nu}$ are fuzzy numbers, therefore $\widetilde{P}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$ will also be fuzzy. Using Zadeh's extension principle [21], the membership function of the performance measure $\widetilde{P}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$ is defined as

$$\mu_{\tilde{P}(\tilde{\lambda},\tilde{\mu},\tilde{\nu})}(z) = \underset{x \in X, y \in Y, w \in W}{Sup} Min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y), \mu_{\tilde{\nu}}(w) / z = P(x, y, w)\right\} \tag{4}$$

Now we derive the performance measures namely the expected number of customers in the orbit and the expected waiting time of the customers in the orbit. Following the work done by Senthilkumar and Arumuganatjan [22] in particular case when the server provides only single phase of service, the expected number of customers in the orbit and the expected waiting time of the customers in the orbit for a crisp retrial queue with batch arrival and Bernoulli vacation schedule for different service and vacation time distributions are as follows:

(a) Model 1: $M^X/E_k/1$ retrial queue with exponential Bernoulli vacation

The expected number of customers in the orbit and the expected waiting time of the customers in the orbit for a crisp retrial queue with batch arrival and Bernoulli vacation schedule for k-Erlangian service and exponential vacation time distributions are given by

$$L_q = \frac{\left[c_1^2 x^2\left(\frac{(k+1)}{ky^2} + \frac{2p}{w^2} + \frac{2p}{yw}\right) + xc_2\left(\frac{1}{y} + \frac{p}{w}\right)\right]}{2\left[1 - c_1 x\left(\frac{1}{y} + \frac{p}{w}\right)\right]} + \frac{\left[c_1 x^2\left(\frac{1}{y} + \frac{p}{w}\right) + x(c_1 - 1)\right]}{\theta\left[1 - c_1 x\left(\frac{1}{y} + \frac{p}{w}\right)\right]} \tag{5}$$

$$W_q = \frac{L_q}{\lambda c_1} \tag{6}$$

(b) Model 2: $M^X/\gamma/1$ retrial queue with k-Erlangian Bernoulli vacation

The expected number of customers in the orbit and the expected waiting time of the customers in the orbit for a crisp retrial queue with batch arrival and Bernoulli vacation schedule for gamma service and k-Erlangian vacation time distributions are given by

$$L_q = \frac{\left[c_1^2 x^2\left(\frac{(k+1)}{ky^2} + \frac{p(k+1)}{kw^2} + \frac{2p}{yw}\right) + xc_2\left(\frac{1}{y} + \frac{p}{w}\right)\right]}{2\left[1 - c_1 x\left(\frac{1}{y} + \frac{p}{w}\right)\right]} + \frac{\left[c_1 x^2\left(\frac{1}{y} + \frac{p}{w}\right) + x\,(c_1 - 1)\right]}{\theta\left[1 - c_1 x\left(\frac{1}{y} + \frac{p}{w}\right)\right]} \tag{7}$$

$$W_q = \frac{L_q}{\lambda c_1} \tag{8}$$

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Using (4), the membership functions for L_q and W_q for model 1 are given by

$$\mu_{\tilde{P}(\tilde{\lambda},\tilde{\mu},\tilde{\nu})}(z) = \underset{x\in X,\, y\in Y,\, w\in W}{Sup}\; Min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y), \mu_{\tilde{\nu}}(w) / z = L_q\right\} \tag{9}$$

$$\mu_{\tilde{P}(\tilde{\lambda},\tilde{\mu},\tilde{\nu})}(z) = \underset{x\in X,\, y\in Y,\, w\in W}{Sup}\; Min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y), \mu_{\tilde{\nu}}(w) / z = W_q\right\} \tag{10}$$

where L_q and W_q are given by eqs (5)-(6).
Similarly, we can compute the membership functions for L_q and W_q for model 2.
Though we have obtained the membership functions for L_q and W_q for the models 1 and 2, but we find difficult to imagine its shape. This problem can be overcome by developing parametric NLPs to find the α-cuts of $\tilde{P}(\tilde{\lambda},\tilde{\mu},\tilde{\nu})$ based on the Zadeh's extension principle which is discussed in the next section.

3.2 The α-cut approach based on extension principle

On the basis of the concept of α-cuts (or α-level sets), we develop a mathematical programming approach for deriving the desired membership function. The definitions for the α-cuts of $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\nu}$ as crisp intervals are as follows (cf. Zimmermann [23]):

$$\lambda_\alpha = \left\{x \in X / \mu_{\tilde{\lambda}}(x) \geq \alpha\right\} \tag{11}$$

$$\mu_\alpha = \left\{y \in Y / \mu_{\tilde{\mu}}(y) \geq \alpha\right\} \tag{12}$$

$$\nu_\alpha = \left\{w \in W / \mu_{\tilde{\nu}}(w) \geq \alpha\right\} \tag{13}$$

It is worthwhile to note that λ_α, μ_α and ν_α are crisp sets rather than fuzzy sets; these crisp sets can be expressed in the following forms:

$$\lambda_\alpha = \left[x_\alpha^L, x_\alpha^U\right] = \left[\underset{x\in X}{Min}\left\{x \in X / \mu_{\tilde{\lambda}}(x) \geq \alpha\right\}, \underset{x\in X}{Max}\left\{x \in X / \mu_{\tilde{\lambda}}(x) \geq \alpha\right\}\right] \tag{14}$$

$$\mu_\alpha = \left[y_\alpha^L, y_\alpha^U\right] = \left[\underset{y\in Y}{Min}\left\{y \in Y / \mu_{\tilde{\mu}}(y) \geq \alpha\right\}, \underset{y\in Y}{Max}\left\{y \in Y / \mu_{\tilde{\mu}}(y) \geq \alpha\right\}\right] \tag{15}$$

$$\nu_\alpha = \left[w_\alpha^L, w_\alpha^U\right] = \left[\underset{w\in W}{Min}\left\{w \in W / \mu_{\tilde{\nu}}(w) \geq \alpha\right\}, \underset{w\in W}{Max}\left\{w \in W / \mu_{\tilde{\nu}}(w) \geq \alpha\right\}\right] \tag{16}$$

The intervals defined above provide information that where the group arrival rate, service rate and vacation rate lie at possibility α. By using the concept of α-cuts, the ***imbedded fuzzy Markov chain*** in the M^X/G/1 retrial queue with Bernoulli vacation schedule, can be decomposed into a family of ordinary Markov chains which possesses different transition probability matrices parameterized by α. The three fuzzy parameters group arrival rate, service rate and vacation rate can also be expressed by different levels of confidence intervals (cf. Negi and Lee [16], Zimmermann [23]). As a result of this, the fuzzy batch arrival retrial queue with Bernoulli vacation can be reduced to a family of crisp batch arrival retrial queue with Bernoulli vacation with different α-level sets $\left\{\lambda_\alpha / 0 < \alpha \leq 1\right\}$, $\left\{\mu_\alpha / 0 < \alpha \leq 1\right\}$ and

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$\{\nu_\alpha / 0 < \alpha \le 1\}$. These three sets represent sets of movable boundaries, forming nested structures for expressing the relationship between ordinary sets and fuzzy sets (cf. Kaufmann [24]). By the convexity of a fuzzy number, the bounds of these intervals are functions of α and can be obtained as

$$x_\alpha^L = Min\, \mu_{\tilde{\lambda}}^{-1}(\alpha),\ x_\alpha^U = Max\, \mu_{\tilde{\lambda}}^{-1}(\alpha)$$
$$y_\alpha^L = Min\, \mu_{\tilde{\mu}}^{-1}(\alpha),\ y_\alpha^U = Max\, \mu_{\tilde{\mu}}^{-1}(\alpha)$$
$$w_\alpha^L = Min\, \mu_{\tilde{\nu}}^{-1}(\alpha),\ w_\alpha^U = Max\, \mu_{\tilde{\nu}}^{-1}(\alpha)$$

Now as defined in eq. (4), the membership function of $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$ is also parameterized by α. Consequently its α-cuts can be used to construct its membership function.

3.3 Construction of membership function

Consider the membership function of L_q. As given in eq. (4), $\mu_{\tilde{L}_q}(z)$ is the minimum of $\mu_{\tilde{\lambda}}(x)$, $\mu_{\tilde{\mu}}(y)$ and $\mu_{\tilde{\nu}}(w)$. To deal with the value of membership function, we need at least one of the following conditions to hold such that $z=L_q$ to satisfy $\mu_{\tilde{L}_q}(z) = \alpha$:

(i) $\mu_{\tilde{\lambda}}(x) = \alpha$, $\mu_{\tilde{\mu}}(y) \ge \alpha$, $\mu_{\tilde{\nu}}(w) \ge \alpha$
(ii) $\mu_{\tilde{\lambda}}(x) \ge \alpha$, $\mu_{\tilde{\mu}}(y) = \alpha$, $\mu_{\tilde{\nu}}(w) \ge \alpha$
(iii) $\mu_{\tilde{\lambda}}(x) \ge \alpha$, $\mu_{\tilde{\mu}}(y) \ge \alpha$, $\mu_{\tilde{\nu}}(w) = \alpha$

Moreover, from the definition of λ_α, μ_α and ν_α given in eqs. (11)-(13), $x \in \lambda_\alpha$, $y \in \mu_\alpha$ and $w \in \nu_\alpha$ can be respectively replaced by $x \in \left[x_\alpha^L, x_\alpha^U\right]$, $y \in \left[y_\alpha^L, y_\alpha^U\right]$ and $w \in \left[w_\alpha^L, w_\alpha^U\right]$. This can be accomplished by using parametric non-linear programming techniques. The NLPs for cases (i) (ii) and (iii) for model 1 are formulated as below:

Case (i):

$$L_{q\alpha}^{L_1} = Min \frac{\left[c_1^2 x^2 \left(\frac{(k+1)}{ky^2} + \frac{2p}{w^2} + \frac{2p}{yw}\right) + x c_2 \left(\frac{1}{y} + \frac{p}{w}\right)\right]}{2\left[1 - c_1 x \left(\frac{1}{y} + \frac{p}{w}\right)\right]} + \frac{\left[c_1 x^2 \left(\frac{1}{y} + \frac{p}{w}\right) + x\,(c_1 - 1)\right]}{\theta\left[1 - c_1 x \left(\frac{1}{y} + \frac{p}{w}\right)\right]}$$

s.t.

$$x_\alpha^L \le x \le x_\alpha^U, \quad y \in \mu_\alpha, w \in \nu_\alpha.$$

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$$L_{q\alpha}^{U_1} = Max \frac{\left[c_1^{\,2}x^{\,2}\left(\frac{(k+1)}{ky^{\,2}}+\frac{2p}{w^{\,2}}+\frac{2p}{yw}\right)+xc_2\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{2\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[c_1x^{\,2}\left(\frac{1}{y}+\frac{p}{w}\right)+x\left(c_1-1\right)\right]}{\theta\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

s.t.

$$x_\alpha^L \le x \le x_\alpha^U,\ y \in \mu_\alpha,\ w \in \nu_\alpha .$$

Case (ii):

$$L_{q\alpha}^{L_2} = Min \frac{\left[c_1^{\,2}x^{\,2}\left(\frac{(k+1)}{ky^{\,2}}+\frac{2p}{w^{\,2}}+\frac{2p}{yw}\right)+xc_2\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{2\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[c_1x^{\,2}\left(\frac{1}{y}+\frac{p}{w}\right)+x\left(c_1-1\right)\right]}{\theta\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

s.t.

$$y_\alpha^L \le y \le y_\alpha^U,\ x \in \lambda_\alpha,\ w \in \nu_\alpha .$$

$$L_{q\alpha}^{U_2} = Min \frac{\left[c_1^{\,2}x^{\,2}\left(\frac{(k+1)}{ky^{\,2}}+\frac{2p}{w^{\,2}}+\frac{2p}{yw}\right)+xc_2\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{2\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[c_1x^{\,2}\left(\frac{1}{y}+\frac{p}{w}\right)+x\left(c_1-1\right)\right]}{\theta\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

s.t.

$$y_\alpha^L \le y \le y_\alpha^U,\ x \in \lambda_\alpha,\ w \in \nu_\alpha .$$

Case (iii):

$$L_{q\alpha}^{L_3} = Min \frac{\left[c_1^{\,2}x^{\,2}\left(\frac{(k+1)}{ky^{\,2}}+\frac{2p}{w^{\,2}}+\frac{2p}{yw}\right)+xc_2\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{2\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[c_1x^{\,2}\left(\frac{1}{y}+\frac{p}{w}\right)+x\left(c_1-1\right)\right]}{\theta\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

s.t.

$$w_\alpha^L \le w \le w_\alpha^U,\ x \in \lambda_\alpha,\ y \in \mu_\alpha .$$

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$$L_{q\alpha}^{U_3} = Min \frac{\left[c_1^2 x^2\left(\frac{(k+1)}{ky^2}+\frac{2p}{w^2}+\frac{2p}{yw}\right)+xc_2\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{2\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[c_1x^2\left(\frac{1}{y}+\frac{p}{w}\right)+x\left(c_1-1\right)\right]}{\theta\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

s.t.

$$w_\alpha^L \le w \le w_\alpha^U,\ x \in \lambda_\alpha,\ y \in \mu_\alpha.$$

The NLPs for cases (i) (ii) and (iii) for the model 2 can be obtained in the same way. In order to find the membership function $\mu_{\tilde{L}_q}(z)$, it is sufficient to find the left shape function (LSF) and the right shape function (RSF) of $\mu_{\tilde{L}_q}(z)$, which in turn are used to find the lower bound $L_{q\alpha}^L$ and the upper bound $L_{q\alpha}^U$ of the α-cuts of $\tilde{L}_q$ for model 1 which can be rewritten as

$$L_{q\alpha}^L = \underset{\substack{x,y,z\in R^+,\\ xc_1<y}}{Min} \frac{\left[c_1^2 x^2\left(\frac{(k+1)}{ky^2}+\frac{2p}{w^2}+\frac{2p}{yw}\right)+xc_2\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{2\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[c_1x^2\left(\frac{1}{y}+\frac{p}{w}\right)+x\left(c_1-1\right)\right]}{\theta\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

s.t.

$$x_\alpha^L \le x \le x_\alpha^U,\ y_\alpha^L \le y \le y_\alpha^U \text{ and } w_\alpha^L \le w \le w_\alpha^U.$$

$$L_{q\alpha}^U = \underset{\substack{x,y,z\in R^+,\\ xc_1<y}}{Max} \frac{\left[c_1^2 x^2\left(\frac{(k+1)}{ky^2}+\frac{2p}{w^2}+\frac{2p}{yw}\right)+xc_2\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{2\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[c_1x^2\left(\frac{1}{y}+\frac{p}{w}\right)+x\left(c_1-1\right)\right]}{\theta\left[1-c_1x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

s.t.

$$x_\alpha^L \le x \le x_\alpha^U,\ y_\alpha^L \le y \le y_\alpha^U \text{ and } w_\alpha^L \le w \le w_\alpha^U.$$

The lower bound and upper bound for model 2 can be written in the same way.

This pair of mathematical programs falls into the category of parametric NLPs which facilitates the systematic study of how the optimal solutions change when x_α^L, x_α^U, y_α^L, y_α^U, w_α^L, w_α^U vary over the interval $\alpha \in (0,1)$.

If both the lower bound $L_{q\alpha}^L$ and the upper bound $L_{q\alpha}^U$ of the α-cuts of $\tilde{L}_q$ are invertible with respect to α, then a LSF and a RSF can be obtained as $L(z) = \left(L_{q\alpha}^L\right)^{-1}$ and $R(z) = \left(L_{q\alpha}^U\right)^{-1}$. Further, the membership function $\mu_{\tilde{L}_q}(z)$ is constructed as

$$\mu_{\tilde{L}_q}(z) = \begin{cases} L(z), & z_1 \le z \le z_2 \\ 1, & z_2 \le z \le z_3 \\ R(z), & z_3 \le z \le z_4 \end{cases} \tag{17}$$

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It is noted that the membership functions for other system characteristics such as expected waiting time of the customers in the orbit for models 1 and 2 can be derived in a similar manner.

4 The numerical illustration

Consider the case of a Mutual Exclusive Life Insurance Company (MELIC) which has launched its branch in a city. There is a single telephone operator which takes care of the incoming calls in its telecommunication department. The group of calls arrives at the company following either Poisson or Gamma distribution. If the telephone line is free, the call is accepted; otherwise the call is stored in a buffer to be received some time later. The probability mass function of the batch size random variable X follows the geometrical distribution with expected value of 2. After each service completion, the telephone operator either goes for vacation such as making phone calls to potential customers to promote the company's service and insurance policies or simply goes for recreation with probability p; or serves the next call with probability (1-p). All the group arrival rate, service rate and vacation rate are trapezoidal fuzzy numbers represented by $\tilde{\lambda} = [2,3,4,5]$, $\tilde{\mu} = [13,14,15,16]$, $\tilde{\nu} = [1,8,15,22]$, respectively. The company manager wishes to determine the average number of calls received by the telephone operator and the average waiting time spend by a call in the buffer due to busy circuits.
Setting c_1=2 and c_2=6 in eqs (5) and (7), we get

(a) For model 1.

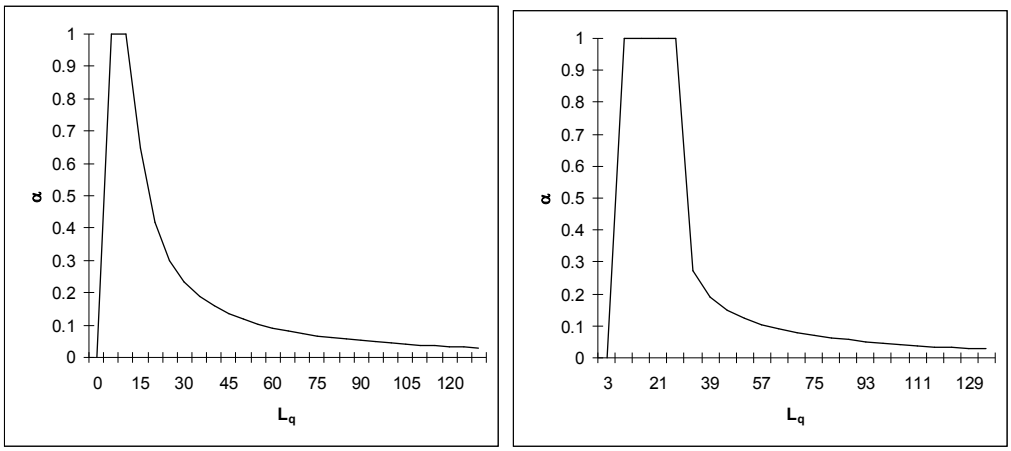
$$L_q = \frac{\left[2x^2\left(\frac{(k+1)}{ky^2}+\frac{2p}{w^2}+\frac{2p}{yw}\right)+3x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{\left[1-2x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[2x^2\left(\frac{1}{y}+\frac{p}{w}\right)+x\right]}{\theta\left[1-2x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

(b) For model 2.

$$L_q = \frac{\left[2x^2\left(\frac{(k+1)}{ky^2}+\frac{p(k+1)}{kw^2}+\frac{2p}{yw}\right)+3x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}{\left[1-2x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}+\frac{\left[2x^2\left(\frac{1}{y}+\frac{p}{w}\right)+x\right]}{\theta\left[1-2x\left(\frac{1}{y}+\frac{p}{w}\right)\right]}$$

We find that $\left[x_\alpha^L, x_\alpha^U\right] = [2+\alpha, 5-\alpha]$, $\left[y_\alpha^L, y_\alpha^U\right] = [13+\alpha, 16-\alpha]$ and $\left[w_\alpha^L, w_\alpha^U\right] = [1+7\alpha, 22-7\alpha]$. As mentioned in section 3, we can formulate the NLPs for deriving the membership functions of $\tilde{L}_q$ and $\tilde{W}_q$ for models 1 and 2.

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(a) **(b)**

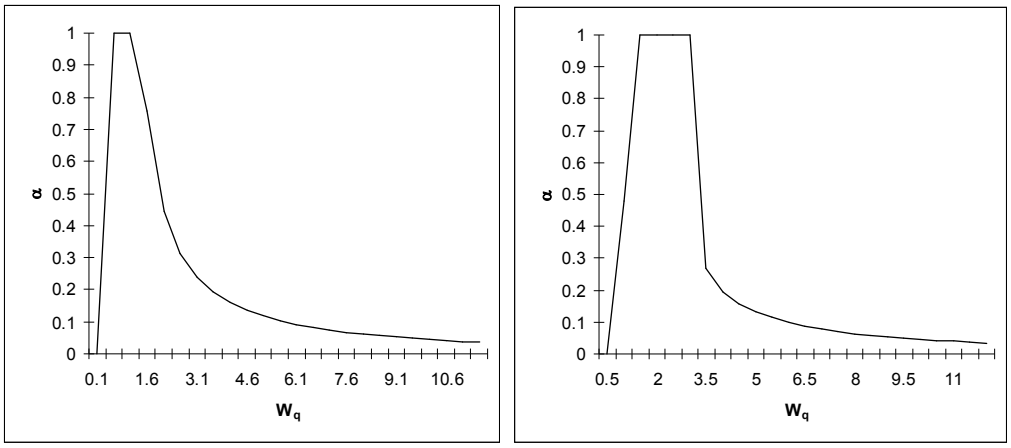
Fig. 1 The membership functions for fuzzy expected number of customers in the orbit for (a) Model 1 (b) Model 2

Then after the upper bounds and lower bounds of α-cuts of L_q and W_q can be obtained for models 1 and 2 as

(a) For model 1.

$$\left(L_q\right)_\alpha^L = \frac{\left(A_1 + B\right)}{D} + \frac{E}{F}; \qquad \left(L_q\right)_\alpha^U = \frac{\left(A_2 + B\right)}{D} + \frac{E}{F}$$

$$\left(W_q\right)_\alpha^L = \frac{\left(L_q\right)_\alpha^L}{2\left(2+\alpha\right)}; \qquad \left(W_q\right)_\alpha^U = \frac{\left(L_q\right)_\alpha^U}{2\left(5-\alpha\right)}$$



(a) **(b)**

Fig. 2 The membership functions for fuzzy expected waiting time of customers in the orbit for (a) Model 1 (b) Model 2

(b) For model 2.

$$\left(L_q\right)_\alpha^L = \frac{\left(A_3 + B\right)}{D} + \frac{E}{F}; \qquad \left(L_q\right)_\alpha^U = \frac{\left(A_4 + B\right)}{D} + \frac{E}{F}$$

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$$\left(W_q\right)_\alpha^L = \frac{\left(L_q\right)_\alpha^L}{2(2+\alpha)}; \qquad \left(W_q\right)_\alpha^U = \frac{\left(L_q\right)_\alpha^U}{2(5-\alpha)}$$

where

$$\begin{aligned}
A_1 &= \left[\{98(k+1)-147k\}+\{32k-21k\}p\right]\alpha^4+\left[\{-224(k+1)+2982k\}+\{-536k+696k\}p\right]\alpha^3\\
&\quad -\left[\{1104(k+1)+9684k\}+\{96k+6012k\}p\right]\alpha^2+\left[\{1408(k+1)-9240\}+\{7072k-1920k\}p\right]\alpha\\
&\quad +\left[\{3872(k+1)+46464k\}+\{9728k+33792k\}p\right],\\
A_2 &= \left[\{98(k+1)-147k\}+\{32k-21k\}p\right]\alpha^4-\left[\{952(k+1)+1218k\}+\{-152k+444k\}p\right]\alpha^3\\
&\quad +\left[\{2172(k+1)+9216k\}-\{3192k+882k\}p\right]\alpha^2+\left[\{680(k+1)+2706\}+\{17628k+4520k\}p\right]\alpha\\
&\quad +\left[\{50(k+1)+195k\}+\{18200k+2535k\}p\right],
\end{aligned}$$

$$\begin{aligned}
A_3 &= \left[98k^2(k+1)+2p(k+1)+28kp\right]\alpha^4-\left[224k^2(k+1)+56p(k+1)+424kp\right]\alpha^3\\
&\quad +\left[-1104k^2(k+1)+264p(k+1)-624kp\right]\alpha^2+\left[1408k^2(k+1)+1792p(k+1)+3488kp\right]\alpha\\
&\quad +\left[3872k^2(k+1)+2048p(k+1)+3632kp\right],\\
A_4 &= \left[98k^2(k+1)+2p(k+1)+28kp\right]\alpha^4+\left[-952k^2(k+1)+32p(k+1)+88kp\right]\alpha^3\\
&\quad +\left[2172k^2(k+1)-132p(k+1)-2928kp\right]\alpha^2+\left[680k^2(k+1)-2080p(k+1)+8680kp\right]\alpha;\\
&\quad +\left[50k^2(k+1)+8450p(k+1)+1300kp\right],\\
B &= -(147+21p)k\alpha^4-(1218+444p)k\alpha^3+(9216-882p)k\alpha^2\\
&\quad +(2706+17628p)k\alpha+(195+2535p),\\
D &= (147+14p)k\alpha^4+(2100+296p)k\alpha^3+(2649+1610p)k\alpha^2\\
&\quad +(588-11752p)k\alpha+(39-1690p),\\
E &= (7+2p)\alpha^3+(-103+6p)\alpha^2+(317-302p)\alpha+(115+1110p),\\
F &= \theta\left[(21+2p)\alpha^2+(24+16p)\alpha+(3-130p)\right].
\end{aligned}$$

Table 1 α-cuts of arrival, service and vacation rates and expected number of customers in the orbit for model 1

α	x_α^L	x_α^U	y_α^L	y_α^U	w_α^L	w_α^U	$\left(L_q\right)_\alpha^L$	$\left(L_q\right)_\alpha^U$
0.0	2.0	5.0	13.0	16.0	1.0	22.0	0.57	343.25
0.1	2.1	4.9	13.1	15.9	1.7	21.3	0.62	56.19
0.2	2.20	4.8	13.20	15.80	2.40	20.60	0.68	33.60
0.3	2.30	4.7	13.30	15.70	3.10	19.90	0.73	25.06
0.4	2.40	4.6	13.40	15.60	3.80	19.20	0.79	20.53
0.5	2.50	4.5	13.50	15.50	4.50	18.50	0.86	17.70
0.6	2.60	4.4	13.60	15.40	5.20	17.80	0.92	15.75
0.7	2.70	4.3	13.70	15.30	5.90	17.10	1.00	14.28
0.8	2.80	4.2	13.80	15.20	6.60	16.40	1.08	13.09
0.9	2.90	4.1	13.90	15.10	7.30	15.70	1.16	12.07
1.0	3.00	4.0	14.00	15.00	8.00	15.00	1.26	11.17

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Table 2 α-cuts of arrival, service and vacation rates and expected number of customers in the orbit for model 2

α	x_α^L	x_α^U	y_α^L	y_α^U	w_α^L	w_α^U	$(L_q)_\alpha^L$	$(L_q)_\alpha^U$
0.0	2.0	5.0	13.0	16.0	1.0	22.0	3.08	344.23
0.1	2.1	4.9	13.1	15.9	1.7	21.3	3.41	58.61
0.2	2.20	4.8	13.20	15.80	2.40	20.60	3.77	38.00
0.3	2.30	4.7	13.30	15.70	3.10	19.90	4.16	31.83
0.4	2.40	4.6	13.40	15.60	3.80	19.20	4.59	29.82
0.5	2.50	4.5	13.50	15.50	4.50	18.50	5.05	29.44
0.6	2.60	4.4	13.60	15.40	5.20	17.80	5.56	29.59
0.7	2.70	4.3	13.70	15.30	5.90	17.10	6.11	29.72
0.8	2.80	4.2	13.80	15.20	6.60	16.40	6.70	29.52
0.9	2.90	4.1	13.90	15.10	7.30	15.70	7.34	28.88
1.0	3.00	4.0	14.00	15.00	8.00	15.00	8.03	27.81

The inverse functions of α-cuts of L_q and W_q can be obtained for models 1 and 2 which in turn give the shape of the membership functions of α-cuts of L_q and W_q for models 1 and 2 as shown in figs 1-2, respectively. Tables 1-4 summarizes the α-cuts of arrival, service and vacation rates, expected number of customers in the orbit and expected waiting time of the customers in the orbit for both models 1 and 2. The default parameters for figs 1-2 and tables 1-4 are chosen as p=0.02, θ=0.8 and k=3. The following observations have been made from these tables and figs:

Table 3 α-cuts of arrival, service and vacation rates and expected waiting time of customers in the orbit for model 1

α	x_α^L	x_α^U	y_α^L	y_α^U	w_α^L	w_α^U	$(W_q)_\alpha^L$	$(W_q)_\alpha^U$
0.0	2.0	5.0	13.0	16.0	1.0	22.0	0.14	34.32
0.1	2.1	4.9	13.1	15.9	1.7	21.3	0.14	5.73
0.2	2.2	4.8	13.2	15.8	2.4	20.6	0.15	3.50
0.3	2.3	4.7	13.3	15.7	3.1	19.9	0.16	2.66
0.4	2.4	4.6	13.4	15.6	3.8	19.2	0.16	2.23
0.5	2.5	4.5	13.5	15.5	4.5	18.5	0.17	1.96
0.6	2.6	4.4	13.6	15.4	5.2	17.8	0.17	1.78
0.7	2.7	4.3	13.7	15.3	5.9	17.1	0.18	1.66
0.8	2.8	4.2	13.8	15.2	6.6	16.4	0.19	1.55
0.9	2.9	4.1	13.9	15.1	7.3	15.7	0.20	1.47
1.0	3.0	4.0	14.0	15.0	8.0	15.0	0.21	1.39

Table 4 α-cuts of arrival, service and vacation rates and expected waiting time of customers in the orbit for model 2

α	x_α^L	x_α^U	y_α^L	y_α^U	w_α^L	w_α^U	$(W_q)_\alpha^L$	$(W_q)_\alpha^U$
0.0	2.0	5.0	13.0	16.0	1.0	22.0	0.77	34.42
0.1	2.1	4.9	13.1	15.9	1.7	21.3	0.81	5.98
0.2	2.2	4.8	13.2	15.8	2.4	20.6	0.85	3.95
0.3	2.3	4.7	13.3	15.7	3.1	19.9	0.90	3.38
0.4	2.4	4.6	13.4	15.6	3.8	19.2	0.95	3.24
0.5	2.5	4.5	13.5	15.5	4.5	18.5	1.01	3.27

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α	x_α^L	x_α^U	y_α^L	y_α^U	w_α^L	w_α^U	$(W_q)_\alpha^L$	$(W_q)_\alpha^U$
0.6	2.6	4.4	13.6	15.4	5.2	17.8	1.06	3.36
0.7	2.7	4.3	13.7	15.3	5.9	17.1	1.13	3.45
0.8	2.8	4.2	13.8	15.2	6.6	16.4	1.19	3.51
0.9	2.9	4.1	13.9	15.1	7.3	15.7	1.26	3.52
1.0	3.0	4.0	14.0	15.0	8.0	15.0	1.33	3.47

(a) For the fuzzy queue length $\tilde{L}_q$, the range of queue length at α=1 is [1.26, 11.17] for model 1 and [8.03, 27.81] for model 2 (cf. tables 1-2) which tell us that it is definitely possible that the number of calls in the buffer falls between 1.26 and 11.17 for model 1 and 8.03 and 27.81 for model 2. Moreover, the range of queue length at α=0 is [0.58, 343.25] for model 1 and [3.08, 344.24] for model 2 (cf. tables 1-2) which indicates that the number of calls in the buffer will never exceed 343.25 (344.24) or fall below 0.58 (3.08) for model 1 (model 2).

(b) For the fuzzy waiting time $\tilde{W}_q$, the range of waiting time in seconds at α=1 is [0.21, 1.39] for model 1 and [1.33, 3.47] for model 2 (cf. tables 3-4) which tell us that it is definitely possible that the waiting time of calls in the buffer falls between 0.21 and 1.39 for model 1 and 1.33 and 3.47 for model 2. Moreover, the range of waiting time in seconds at α=0 is [0.14, 34.32] for model 1 and [0.77, 34.42] for model 2 (cf. tables 3-4); it indicates that the waiting time of calls in the buffer will never exceed 34.32 (34.42) or fall below 0.14 (0.77) for model 1 (model 2).

(c) The information observed from the numerical investigation will be very useful for designing a queuing system which involves one or more combinations of several decisions, such as the efficiency of the servers, number of servers, waiting time of the customers, etc..

5 Conclusions

In this paper, we have used the α-cut approach to analyze a fuzzy queuing model. A pair of parametric NLPs to find the α-cuts of the membership functions of the performance measures is employed. Following the proposed approach, α-cuts of the membership functions are found and to attain explicit closed-form expressions for the system characteristics, their interval limits are inverted. As the performance measures are expressed by membership functions rather than by crisp values, more information provided to the system designers and decision makers may be helpful to improve the existing systems. Since the fuzzy performance measures of fuzzy queues derived from the proposed approach maintain the fuzziness of input information; therefore the derived results can be used to represent the real time systems as fuzzy system more accurately.

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