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Malmquist productivity index in several time periods on interval data

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Abstract The productivity change of a decision making unit (DMU) between two time periods can be evaluated by the Malmquist productivity index. In this study, we propose a method to compute the Malmquist productivity index in several time periods (from the first to the last periods) on interval data in data envelopment analysis (DEA). Then, the obtained Malmquist productivity index is compared with Malmquist productivity index between two time periods (the first and last time periods) on interval data. The aim of this paper is to investigate the progress and regress of decision making units (DMUs) in several time periods by considering all time periods between the first and the last one, on interval data. Consequently when the Malmquist productivity index is computed in several time periods, progress and regress of decision making units can be evaluated more carefully than before. What is more, one can judge with greater accuracy whether or not such strategy are favorable and promising. Lastly, a numerical demonstration illustrates the procedure of the proposed method; then some conclusions are reached, and directions for future research are suggested.

Keywords Data Envelopment Analysis (DEA), Efficiency, Malmquist Productivity Index (MPI), Interval Data.

1 Introduction

The Malmquist index was first suggested by Malmquist [1] as a quantity index using in the analysis of consumption of inputs. A DEA-based Malmquist productivity index was developed by Färe et al. [2] that it measures the productivity change over time. These ideas were combined on the measurement of efficiency from Farrell and the measurement of efficiency from Caves et al. [3] by Färe et al. for constructing a Malmquist productivity index.

It has been proved that the Malmquist productivity index can be a good tool for measuring the productivity change of decision making units DMUs. Some researchers have already paid attention to the measurement of the productivity change of DMUs [4-9] and also, there are some researches for performance evaluation of DMUs [10-26].

In recent years, the Malmquist productivity index (MPI) is very useful for calculating the productivity change a DMU that extensively used in the management science literature. It is especially useful for examining the effect of an act or process. For instance, Pascoe and Herrero [27] applied the Malmquist index (MI) in order to calculate a stock index based on

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the changes in the DEA efficiency scores over time. This method was applied to two Spanish fisheries operating in the South Atlantic-a single- species fishery and a multi-species fishery. Besides, MI is based on multi input-output frontier representations of the productivity technology.

So far the Malmquist productivity index has been computed between two time periods for assessing the productivity change of DMUs. In this vein, Hosseinzadeh Lotfi et al. [28] computed the Malmquist productivity index between two time periods on interval data for evaluating the productivity change of DMUs. Again, Koa [29] investigated Malmquist productivity index based on common-weights in DEA. Subsequently, deriving the DEA frontier for two-stage processes has been inspected by Chen et al. [30].

Furthermore, Pastor et al. [31] introduced the biennial Malmquist productivity change index. Later, a new approach based on double frontiers data envelopment analysis for measuring Malmquist productivity index has been presented by Wang and Lan [32].

The objective of this study is to design a MPI, which is based on computing the MPI in several time periods on interval data for evaluating the productivity change of DMUs.

This article is organized as follows. Section 2 briefly describes the Malmquist productivity index on interval data. The proposed method is described in Section 3. In Section 4, an empirical example is provided to illustrate our proposed method. Lastly, a conclusion comment and future extensions are summarized in Section 5.

2 Preliminaries

The Malmquist productivity index is defined as the product of $Catch-up$ and $Frontier-shif$ terms for evaluating the productivity change of a DMU between two time periods. The catch-up term Relates the efficiency change of a DMU, while the frontier-shift term reflects the change in the efficient frontiers over two time periods.

Consider n DMUs, DMU_j $(j=1,2,\ldots,n)$, that each having m inputs denoted by x_{ij} $(x_{ij}>0,\ i=1,2,\ldots,m)$ and s outputs denoted by y_{rj} $(y_{rj}>0,\ r=1,2,\ldots,s)$ over the periods t and $t+1$. Let's assume that the levels of inputs and outputs are known to lie within the bounded intervals, i.e., $x_{ij}^{l}\in\left[x_{ij}^{l,L},x_{ij}^{l,U}\right], i=t,t+1; i=1,\ldots,m; j=1,\ldots,n$ and $y_{rj}^{l}\in\left[y_{rj}^{l,L},y_{rj}^{l,U}\right], i=t,t+1; r=1,\ldots,s; j=1,\ldots,n$. It is necessary to mention that the upper and lower bounds of intervals are constant and strictly positive. We also use the notations $(x_{1o}^{t},\ldots,x_{mo}^{t},y_{1o}^{t},\ldots,y_{so}^{t})=(\mathbf{x}_o^{t},\mathbf{y}_o^{t})=(\mathbf{x}_o,\mathbf{y}_o)^{t}$ and $(x_{1o}^{t+1},\ldots,x_{mo}^{t+1},y_{1o}^{t+1},\ldots,y_{so}^{t+1})=(\mathbf{x}_o^{t+1},\mathbf{y}_o^{t+1})=(\mathbf{x}_o,\mathbf{y}_o)^{t+1}$ for representing DMU_o, $\left(o\in\{1,2,\ldots,n\}\right)$ in periods t and $t+1$, respectively. The production possibility set (PPS) T^{l} $(l=t$ and $t+1)$ is defined as follows:

$$T^{l}=\left\{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{m+s}\ \middle|\ \mathbf{x}\geq\sum_{j=1}^{n}\lambda_j\left[\mathbf{x}_j^{l,L},\mathbf{x}_j^{l,U}\right],\ \mathbf{0}\leq\mathbf{y}\leq\sum_{j=1}^{n}\lambda_j\left[\mathbf{y}_j^{l,L},\mathbf{y}_j^{l,U}\right],\ K\leq\sum_{j=1}^{n}\lambda_j\leq H,\ \lambda_j\geq 0\ (j=1,2,\ldots,n)\right\} \quad (1)$$

where $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbb{R}^{n}$ is the intensity vector. $(K,H)=\{(0,\infty),(1,1),(1,\infty),$ and $(0,1)\}$ corresponds to the CCR, BCC, IRS and DRS models, respectively.

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Since data is inexact, and they lie within the bounded intervals so, the catch-up and frontier-shift effects change as an interval. The upper and lower bounds of the catch-up effect over two time periods t and $t+1$ are respectively computed by following formulas.

$$(Catch-up)^U = \frac{{}^o\theta_{t+1}^{t+1,U}}{{}^o\theta_t^{t,L}}, \tag{2}$$

$$(Catch-up)^L = \frac{{}^o\theta_{t+1}^{t+1,L}}{{}^o\theta_t^{t,U}}, \tag{3}$$

where ${}^o\theta_{t+1}^{t+1,U}$ and ${}^o\theta_{t+1}^{t+1,L}$ are respectively the upper and lower bounds of the efficiency of $(\mathbf{x}_o, \mathbf{y}_o)^{t+1}$ with respect to frontier of period $t+1$ [14]. Also, ${}^o\theta_t^{t,U}$ and ${}^o\theta_t^{t,L}$ are the upper and lower bounds of the efficiency of $(\mathbf{x}_o, \mathbf{y}_o)^t$ with respect to frontier of period t, respectively [14].

For simplicity matters, Fig. 1 depicts the case of a single input and output $(m = s = 1)$ [12].

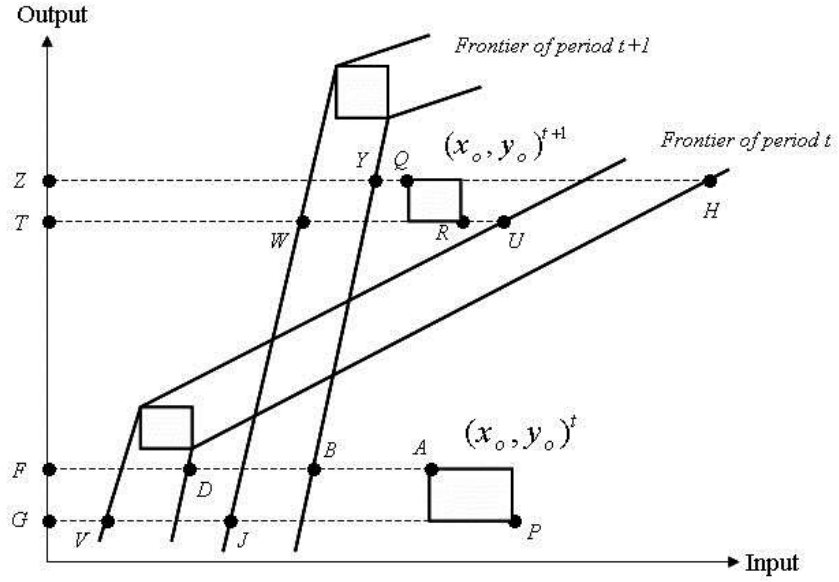


Fig. 1 Two time periods

The upper and lower bounds of the catch-up effect are respectively computed in an input-orientation as follows:

$$(Catch-up)^U = \frac{ZY}{ZQ} \Big/ \frac{GV}{GP}, \tag{4}$$

$$(Catch-up)^L = \frac{TW}{TR} \Big/ \frac{FD}{FA}. \tag{5}$$

Moreover, the upper and lower bounds of the frontier-shift effect are respectively computed by the following geometric means:

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$$(Frontier-shift)^U = \sqrt{\varphi_1^U \varphi_2^U}, \tag{6}$$

$$(Frontier-shift)^L = \sqrt{\varphi_1^L \varphi_2^L}, \tag{7}$$

where $\varphi_1^U = \frac{{}^o\theta_t^{t,U}}{{}^o\theta_t^{t+1,L}}$ and $\varphi_1^L = \frac{{}^o\theta_t^{t,L}}{{}^o\theta_t^{t+1,U}}$ are respectively the upper and lower bounds of the frontier-shift effect at $(\mathbf{x}_o, \mathbf{y}_o)^t$ and also, $\varphi_2^U = \frac{{}^o\theta_{t+1}^{t,U}}{{}^o\theta_{t+1}^{t+1,L}}$ and $\varphi_2^L = \frac{{}^o\theta_{t+1}^{t,L}}{{}^o\theta_{t+1}^{t+1,U}}$ are the upper and lower bounds of the frontier-shift effect at $(\mathbf{x}_o, \mathbf{y}_o)^{t+1}$, respectively. Note that, ${}^o\theta_t^{t+1,U}$ and ${}^o\theta_t^{t+1,L}$ are respectively the upper and lower bounds of the efficiency of $(\mathbf{x}_o, \mathbf{y}_o)^t$ with respect to frontier of period $t+1$ [14]. Besides, ${}^o\theta_{t+1}^{t,U}$ and ${}^o\theta_{t+1}^{t,L}$ are the upper and lower bounds of the efficiency of $(\mathbf{x}_o, \mathbf{y}_o)^{t+1}$ with respect to frontier of period t, respectively [14].

According to Fig. 1, the upper and lower bounds of the frontier-shift effect are computed as follows:

$$(Frontier-shift)^U = \left(\frac{FD \times GP}{FA \times GJ} \times \frac{ZH \times TR}{ZQ \times TW}\right)^{\frac{1}{2}}, \tag{8}$$

$$(Frontier-shift)^L = \left(\frac{GV \times FA}{GP \times FB} \times \frac{TV \times ZQ}{TR \times ZY}\right)^{\frac{1}{2}}, \tag{9}$$

where $\varphi_1^U = \frac{FD}{FA} \Big/ \frac{GJ}{GP}$, $\varphi_1^L = \frac{GV}{GP} \Big/ \frac{FB}{FA}$, $\varphi_2^U = \frac{ZH}{ZQ} \Big/ \frac{TW}{TR}$, and $\varphi_2^L = \frac{TU}{TR} \Big/ \frac{ZY}{ZQ}$.

Since data lies within the bounded intervals hence, the Malmquist index (MI) change as an interval. The upper bound of the Malmquist index (MI^U) is computed as the product of $(Catch-up)^U$ and $(Frontier-shift)^U$ also, the lower bound of the Malmquist index (MI^L) is computed as the product of $(Catch-up)^L$ and $(Frontier-shift)^L$, i.e.,

$$MI^U = (Catch-up)^U \times (Frontier-shift)^U, \tag{10}$$

$$MI^L = (Catch-up)^L \times (Frontier-shift)^L. \tag{11}$$

Thus, using (2), (3), (6), and (7), the upper and lower bounds of the Malmquist index for evaluating change of DMU_o are as follows:

$$\begin{aligned} MI^U = MI^{o,U} &= (Catch-up)^{o,U} \times (Frontier-shift)^{o,U}, \\ &= \frac{{}^o\theta_{t+1}^{t+1,U}}{{}^o\theta_t^{t,L}} \left(\frac{{}^o\theta_t^{t,U}}{{}^o\theta_t^{t+1,L}} \times \frac{{}^o\theta_{t+1}^{t,U}}{{}^o\theta_{t+1}^{t+1,L}}\right)^{\frac{1}{2}}, \end{aligned} \tag{12}$$

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$$MI^{L} = MI^{o,L} = (Catch-up)^{o,L} \times (Frontier-shift)^{o,L},$$
$$= \frac{{}^{o}\theta_{t+1}^{t+1,L}}{{}^{o}\theta_{t}^{t,U}} \left(\frac{{}^{o}\theta_{t}^{t,L}}{{}^{o}\theta_{t}^{t+1,U}} \times \frac{{}^{o}\theta_{t+1}^{t,L}}{{}^{o}\theta_{t+1}^{t+1,U}} \right)^{\frac{1}{2}}, \tag{13}$$

where the upper and lower bounds of the relative change in performance are respected by the first terms and the second terms respect the upper and lower bounds of the relative change in the frontier used to evaluate these performances.

According to Fig. 1, the upper and lower bounds of the Malmquist index are computed as:

$$MI^{o,U} = \frac{ZY \times GP}{ZQ \times GV} \left(\frac{FD \times GP}{FA \times GJ} \times \frac{ZH \times TR}{ZQ \times TW} \right)^{\frac{1}{2}}, \tag{14}$$

$$MI^{o,L} = \frac{TW \times FA}{TR \times FD} \left(\frac{GV \times FA}{GP \times FB} \times \frac{TV \times ZQ}{TR \times ZY} \right)^{\frac{1}{2}}. \tag{15}$$

It is noticeable that, $MI^{o,L} > 1$ and $MI^{o,U} < 1$ indicate progress and regress for DMU_o over two time periods t and $t+1$, respectively otherwise, progress and regress of DMU_o are evaluated between two time periods t and $t+1$ as follows.

(a) If $MI^{o,L} = 1$ and $MI^{o,U} = 1$, then there is no progress and no regress for DMU_o.
(b) If $MI^{o,L} = 1$ and $MI^{o,U} > 1$, then there is progress for DMU_o.
(c) If $MI^{o,L} < 1$ and $MI^{o,U} = 1$, then there is regress for DMU_o.
(d) If $MI^{o,L} < 1$ and $MI^{o,U} > 1$, then an index is presented as follows:

$$\mu^{o} = \frac{MI^{o,U} - 1}{1 - MI^{o,L}}. \tag{16}$$

It is clear that $0 < \mu^{o} < +\infty$. Note that, $\mu^{o} > 1$ indicates more percent of the progress with respect to regress for DMU_o and also, $\mu^{o} < 1$ indicates more percent of the regress with respect to progress for it.

3 New insights from Malmquist productivity approach

In this section, we propose a method to compute the Malmquist productivity index in p ($p \geq 3$) time periods on interval data for evaluating the productivity change of a DMU. On the other hand, we will compute the Malmquist productivity index from period t to $t+p-1$ on interval data. The notation $(x_{1o}^{t+i}, \ldots, x_{mo}^{t+i}, y_{1o}^{t+i}, \ldots, y_{so}^{t+i}) = (\mathbf{x}_o, \mathbf{y}_o)^{t+i}$ $(i = 0,1,\ldots,p-1)$ is

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used for representing DMU_o in period $t+i$, where $x_{ko}^{t+i} \in \left[x_{ko}^{t+i,L}, x_{ko}^{t+i,U} \right], k = 1,\ldots,m$ and $y_{ro}^{t+i} \in \left[y_{ro}^{t+i,L}, y_{ro}^{t+i,U} \right], r = 1,\ldots,s$.

In this method, the upper and lower bounds of the Malmquist index are first computed for evaluating productivity change of DMU_o between two time periods $t+i$ and $t+i+1$ $(i = 0,1,\ldots,p-2)$ by using (12) and (13). Here, they are represented by notations $MI_{i+1}^{o,U}$ and $MI_{i+1}^{o,L}$, respectively. Then, we compute the upper and lower bounds of the Malmquist index for evaluating productivity change of DMU_o over p time periods as the product of $MI_{i+1}^{o,U}$ $(i = 0,1,\ldots,p-2)$ and the product of $MI_{i+1}^{o,L}$ $(i = 0,1,\ldots,p-2)$, respectively, i.e.,

$$\begin{aligned} MI_{total}^{U} = MI_{total}^{o,U} &= (Catch-up)_{total}^{o,U} \times (Frontier-shift)_{total}^{o,U}, \\ &= \prod_{i=0}^{p-2} MI_{i+1}^{o,U}, \end{aligned} \tag{17}$$

$$\begin{aligned} MI_{total}^{L} = MI_{total}^{o,L} &= (Catch-up)_{total}^{o,L} \times (Frontier-shift)_{total}^{o,L}, \\ &= \prod_{i=0}^{p-2} MI_{i+1}^{o,L}, \end{aligned} \tag{18}$$

because the performance of DMU_o between each two consecutive time periods does not depend on its performance between each two another consecutive time periods.

It is note worthy that, $(Catch-up)_{total}^{o,U}$ and $(Catch-up)_{total}^{o,L}$ are the upper and lower bounds of the catch-up effect of DMU_o from period t to $t+p-1$, respectively. They can be computed as follows:

$$(Catch-up)_{total}^{o,U} = \prod_{i=0}^{p-2} (Catch-up)_{i+1}^{o,U}, \tag{19}$$

$$(Catch-up)_{total}^{o,L} = \prod_{i=0}^{p-2} (Catch-up)_{i+1}^{o,L}. \tag{20}$$

In addition, $(Frontier-shift)_{total}^{o,U}$ and $(Frontier-shift)_{total}^{o,L}$ are respectively the upper and lower bounds of the frontier-shirt effect of DMU_o from period t to $t+p-1$. Here, they can be calculated as follows:

$$(Frontier-shift)_{total}^{o,U} = \prod_{i=0}^{p-2} (Frontier-shift)_{i+1}^{o,U}, \tag{21}$$

$$(Frontier-shift)_{total}^{o,L} = \prod_{i=0}^{p-2} (Frontier-shift)_{i+1}^{o,L}. \tag{22}$$

In the case of a single input and output, a simple example is presented in Fig. 2 [12].

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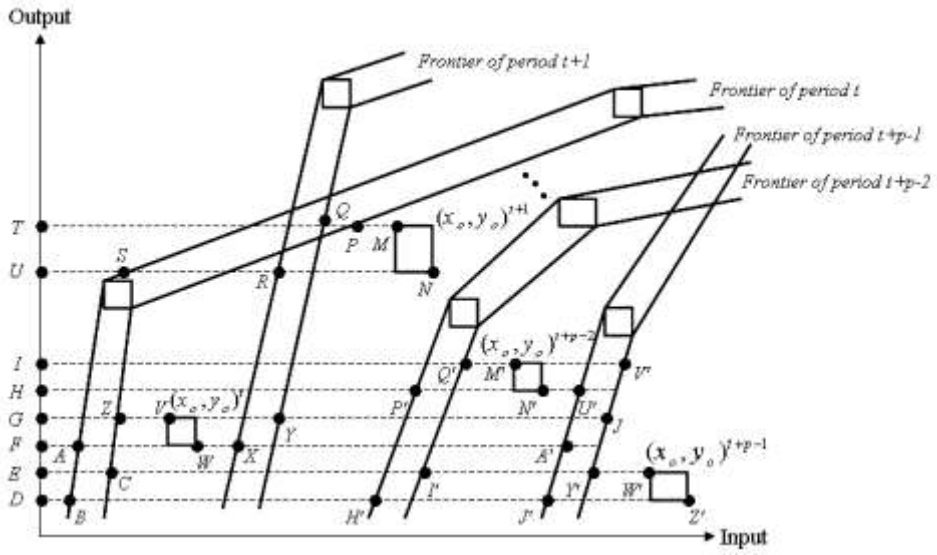


Fig. 2 Several time periods

In what follows, using (12), (13), (17), and (18), we get

$$MI_{total}^{o,U} = MI_1^{o,U} \times \ldots \times MI_{p-1}^{o,U},$$
$$= \frac{TQ \times FW}{TM \times FA}\left(\frac{GZ \times FW}{GV \times FX} \times \frac{TP \times UN}{TM \times UR}\right)^{\frac{1}{2}} \times \ldots \times \frac{EY' \times HN'}{EW' \times HP'}\left(\frac{IQ' \times HN'}{IM' \times HU'} \times \frac{EI' \times DZ'}{EW' \times DJ'}\right)^{\frac{1}{2}}, \quad (23)$$

$$MI_{total}^{o,L} = MI_1^{o,L} \times \ldots \times MI_{p-1}^{o,L},$$
$$= \frac{UR \times GV}{UN \times GZ}\left(\frac{FA \times GV}{FW \times GY} \times \frac{US \times TM}{UN \times TQ}\right)^{\frac{1}{2}} \times \ldots \times \frac{DJ' \times IM'}{DZ' \times IQ'}\left(\frac{HP' \times IM'}{HN' \times IV'} \times \frac{DH' \times EW'}{DZ' \times EY'}\right)^{\frac{1}{2}}. \quad (24)$$

Now, using (12) and (13), we compute the upper and lower bounds of the Malmquist index for evaluating productivity change of DMU_o over two time periods t and $t+p-1$ that as they are denoted by notations $MI_{1,p}^{o,U}$ and $MI_{1,p}^{o,L}$, respectively, as bellows:

$$MI_{1,p}^{U} = MI_{1,p}^{o,U} = (Catch-up)_{1,p}^{o,U} \times (Frontier-shift)_{1,p}^{o,U}, \quad (25)$$
$$MI_{1,p}^{L} = MI_{1,p}^{o,L} = (Catch-up)_{1,p}^{o,L} \times (Frontier-shift)_{1,p}^{o,L}, \quad (26)$$

where $(Catch-up)_{1,p}^{o,U}$ and $(Catch-up)_{1,p}^{o,L}$ are the upper and lower bounds of the catch-up effect of DMU_o over two time periods t and $t+p-1$, respectively and also, $(Frontier-shift)_{1,p}^{o,U}$ and $(Frontier-shift)_{1,p}^{o,L}$ are respectively the upper and lower bounds of the frontier-shift effect of DMU_o between two time periods t and $t+p-1$.

Associated with Fig. 2, $MI_{1,p}^{o,U}$ and $MI_{1,p}^{o,L}$ can be computed as follows:

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$$MI_{1,p}^{o,U} = \frac{EY' \times FW}{EW' \times FA}\left(\frac{GZ \times FW}{GV \times FA'} \times \frac{EC \times DZ'}{EW' \times DJ'}\right)^{\frac{1}{2}}, \tag{27}$$

$$MI_{1,p}^{o,l} = \frac{DJ' \times GV}{DZ' \times GZ}\left(\frac{FA \times GV}{FW \times GJ} \times \frac{DB \times EW'}{DZ' \times EY'}\right)^{\frac{1}{2}}. \tag{28}$$

Theorem 1. *The upper and lower bounds of catch-up effect of* DMU_o *between two time periods* t *and* $t+p-1$ *equal its upper and lower bounds of the catch-up effect from period* t *to* $t+p-1$*, respectively, i.e.,*

$$(Catch-up)_{1,p}^{o,U} = (Catch-up)_{total}^{o,U}, \tag{29}$$

$$(Catch-up)_{1,p}^{o,L} = (Catch-up)_{total}^{o,L}. \tag{30}$$

Proof. The proof is straightforward from (2) and (3). □

Theorem 2. *The relation between* $(Frontier-shift)_{total}^{o,U}$ *and* $(Frontier-shift)_{1,p}^{o,U}$ *is as follows:*

$$(Frontier-shift)_{total}^{o,U} = \left(\frac{{}^o\theta_t^{t+p-1,L}}{{}^o\theta_{t+p-1}^{t,U}} \times \frac{{}^o\theta_{t+1}^{t,U}}{{}^o\theta_{t+p-2}^{t+p-1,L}} \times \frac{\prod_{i=1}^{p-2}\left({}^o\theta_{t+i}^{t+i,U} \times {}^o\theta_{t+i+1}^{t+i,U}\right)}{\prod_{i=0}^{p-3}\left({}^o\theta_{t+i+1}^{t+i+1,L} \times {}^o\theta_{t+i}^{t+i+1,L}\right)}\right)^{\frac{1}{2}} \times (Frontier-shift)_{1,p}^{o,U}, \tag{31}$$

and also, the relation between $(Frontier-shift)_{total}^{o,L}$ and $(Frontier-shift)_{1,p}^{o,L}$ is as:

$$(Frontier-shift)_{total}^{o,L} = \left(\frac{{}^o\theta_t^{t+p-1,U}}{{}^o\theta_{t+p-1}^{t,L}} \times \frac{{}^o\theta_{t+1}^{t,L}}{{}^o\theta_{t+p-2}^{t+p-1,U}} \times \frac{\prod_{i=1}^{p-2}\left({}^o\theta_{t+i}^{t+i,L} \times {}^o\theta_{t+i+1}^{t+i,L}\right)}{\prod_{i=0}^{p-3}\left({}^o\theta_{t+i+1}^{t+i+1,U} \times {}^o\theta_{t+i}^{t+i+1,U}\right)}\right)^{\frac{1}{2}} \times (Frontier-shift)_{1,p}^{o,L}. \tag{32}$$

Proof. According to (6) and (21), the upper bound of the frontier-shift effect of DMU_o from period t to $t+p-1$ is as follows:

$$(Frontier-shift)_{total}^{o,U} = \left(\frac{\prod_{i=0}^{p-2}\left({}^o\theta_{t+i}^{t+i,U} \times {}^o\theta_{t+i+1}^{t+i,U}\right)}{\prod_{i=0}^{p-2}\left({}^o\theta_{t+i+1}^{t+i+1,L} \times {}^o\theta_{t+i}^{t+i+1,L}\right)}\right)^{\frac{1}{2}}. \tag{33}$$

Thus,

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$$(Frontier-shift)_{total}^{o,U}=\left(\frac{{}^{o}\theta_{t}^{t,U}}{{}^{o}\theta_{t+p-1}^{t+p-1,L}}\times\frac{{}^{o}\theta_{t+1}^{t,U}}{{}^{o}\theta_{t+p-2}^{t+p-1,L}}\times\frac{{}^{o}\theta_{t+p-1}^{t,U}}{{}^{o}\theta_{t}^{t+p-1,L}}\times\frac{{}^{o}\theta_{t}^{t+p-1,L}}{{}^{o}\theta_{t+p-1}^{t,U}}\times\frac{\prod_{i=1}^{p-2}\left({}^{o}\theta_{t+i}^{t+i,U}\times{}^{o}\theta_{t+i+1}^{t+i,U}\right)}{\prod_{i=0}^{p-3}\left({}^{o}\theta_{t+i+1}^{t+i+1,L}\times{}^{o}\theta_{t+i}^{t+i+1,L}\right)}\right)^{\frac{1}{2}},$$

$$=\left(\frac{{}^{o}\theta_{t}^{t+p-1,L}}{{}^{o}\theta_{t+p-1}^{t,U}}\times\frac{{}^{o}\theta_{t+1}^{t,U}}{{}^{o}\theta_{t+p-2}^{t+p-1,L}}\times\frac{\prod_{i=1}^{p-2}\left({}^{o}\theta_{t+i}^{t+i,U}\times{}^{o}\theta_{t+i+1}^{t+i,U}\right)}{\prod_{i=0}^{p-3}\left({}^{o}\theta_{t+i+1}^{t+i+1,L}\times{}^{o}\theta_{t+i}^{t+i+1,L}\right)}\right)^{\frac{1}{2}}\times(Frontier-shift)_{1,p}^{o,U},\tag{34}$$

and the proof is complete. (32) is similarly proved. □

Now, we first compute $MI_{total}^{o,U}$, $MI_{total}^{o,L}$, $MI_{1,p}^{o,L}$, and $MI_{1,p}^{o,U}$, then $\left[MI_{total}^{o,L},MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L},MI_{1,p}^{o,U}\right]$ are compare to evaluate progress and regress of DMU_o from period t to $t+p-1$ as follows:

(1) If $MI_{total}^{o,L}<MI_{total}^{o,U}<MI_{1,p}^{o,L}<MI_{1,p}^{o,U}$, then we will have the below cases:

- **(1-1)** If $MI_{1,p}^{o,U}\le 1$, then both of $\left[MI_{total}^{o,L},MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L},MI_{1,p}^{o,U}\right]$ indicate regress for DMU_o and also, $\left[MI_{total}^{o,L},MI_{total}^{o,U}\right]$ indicates more regress for it.
- **(1-2)** If $MI_{1,p}^{o,L}<1$ and $MI_{1,p}^{o,U}>1$, then according to (16), $\mu^o>1$ indicates more percent of the progress with respect to regress for DMU_o and also, $\mu^o<1$ indicates more percent of the regress with respect to progress for it, while $\left[MI_{total}^{o,L},MI_{total}^{o,U}\right]$ perfectly indicates regress for DMU_o.
- **(1-3)** If $MI_{1,p}^{o,L}=1$, then $\left[MI_{total}^{o,L},MI_{total}^{o,U}\right]$ indicates regress for DMU_o, while $\left[MI_{1,p}^{o,L},MI_{1,p}^{o,U}\right]$ indicates progress for it.
- **(1-4)** If $MI_{total}^{o,U}\le 1$ and $MI_{1,p}^{o,L}>1$, then $\left[MI_{total}^{o,L},MI_{total}^{o,U}\right]$ indicates regress for DMU_o, while $\left[MI_{1,p}^{o,L},MI_{1,p}^{o,U}\right]$ indicates progress for it.
- **(1-5)** If $MI_{total}^{o,L}>1$ and $MI_{total}^{o,U}<1$, then we present an index as follows:

$$\delta^o=\frac{MI_{total}^{o,U}-1}{1-MI_{total}^{o,L}}.\tag{35}$$

It is obvious that $0<\delta^o<+\infty$. Note that, $\delta^o>1$ indicates more percent of the progress with respect to regress for DMU_o and also, $\delta^o<1$ indicates more percent of the regress with respect to progress for it, while $\left[MI_{1,p}^{o,L},MI_{1,p}^{o,U}\right]$ perfectly indicates progress for DMU_o.

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(1-6) If $MI_{total}^{o,L} \geq 1$, then both of $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ indicate progress for DMU_o and also, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates less progress for it.

(2) If $MI_{total}^{o,L} < MI_{1,p}^{o,L} < MI_{total}^{o,U} < MI_{1,p}^{o,U}$, then we will have the below cases:

(2-1) If $MI_{1,p}^{o,U} \leq 1$, then both of $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ indicate regress for DMU_o and also, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more regress for it.

(2-2) If $MI_{total}^{o,U} \leq 1$ and $MI_{1,p}^{o,U} > 1$, then according to (16), progress and regress of the DMU_o are evaluated as like as (1-2), while $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ perfectly indicates regress for DMU_o.

(2-3) If $MI_{1,p}^{o,L} < 1$ and $MI_{total}^{o,U} > 1$, then according to (16) and (35), $\delta^o < \mu^o$ and progress and regress of the DMU_o are evaluated as like as (1-2) and (1-5). Also, according to $\delta^o < \mu^o$, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more percent of the regress and less percent of the progress for DMU_o.

(2-4) If $MI_{total}^{o,L} < 1$ and $MI_{1,p}^{o,L} \geq 1$, then according to (35), progress and regress of the DMU_o are evaluated as like as (1-5), while $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ perfectly indicates progress for DMU_o.

(2-5) If $MI_{total}^{o,L} \geq 1$, then both of $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ indicate progress for DMU_o and also, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates less progress for it.

(3) If $MI_{1,p}^{o,L} = MI_{total}^{o,L} < MI_{1,p}^{o,U} < MI_{total}^{o,U}$, then we will have the below cases:

(3-1) If $MI_{total}^{o,U} \leq 1$, then both of $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ indicate regress for DMU_o and also, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates less regress for it.

(3-2) If $MI_{1,p}^{o,U} \leq 1$ and $MI_{total}^{o,U} > 1$, then according to (35), progress and regress of the DMU_o are evaluated as like as (1-5), while $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ perfectly indicates regress for DMU_o.

(3-3) If $MI_{total}^{o,L} < 1$ and $MI_{1,p}^{o,U} > 1$, then according to (16) and (35), $\delta^o > \mu^o$ and progress and regress of the DMU_o are evaluated as like as (1-2) and (1-5). Also, according to $\delta^o > \mu^o$, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more percent of the progress for DMU_o.

(3-4) If $MI_{total}^{o,L} \geq 1$, then both of $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ indicate progress for DMU_o and also, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more progress for it.

(4) If $MI_{1,p}^{o,L} < MI_{total}^{o,L} < MI_{total}^{o,U} < MI_{1,p}^{o,U}$, then we will have the below cases:

(4-1) If $MI_{1,p}^{o,U} \leq 1$, then both of $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ indicate regress for DMU_o and also, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more regress for it.

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(4-2) If $MI_{total}^{o,U} \le 1$ and $MI_{1,p}^{o,U} > 1$, then according to (16), progress and regress of the DMU_o are evaluated as like as (1-2), while $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ perfectly indicates regress for DMU_o.

(4-3) If $MI_{total}^{o,L} < 1$ and $MI_{total}^{o,U} > 1$, then according to (16) and (35), progress and regress of the DMU_o are evaluated as like as (1-2) and (1-5), respectively and we will have the below cases:

(4-3-1) If $\delta^o < \mu^o$, then $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more percent of regress and less percent of progress for DMU_o.

(4-3-2) If $\delta^o > \mu^o$, then $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more percent of progress and less percent of regress for DMU_o.

(4-3-3) If $\delta^o = \mu^o$, then the obtained results from (1-2) and (1-5) are the same.

(4-4) If $MI_{total}^{o,L} \ge 1$ and $MI_{1,p}^{o,L} < 1$, then according to (16), progress and regress of the DMU_o are evaluated as like as (1-2), while $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ perfectly indicates progress for DMU_o.

(4-5) If $MI_{1,p}^{o,L} \ge 1$, then both of $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ and $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$ indicate progress for DMU_o and also, $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ indicates more progress for it.

Note that, four cases were discussed above and the other cases can be similarly discussed.

According to the above discussion, it is clear that the obtained results from $\left[MI_{total}^{o,L}, MI_{total}^{o,U}\right]$ about progress and regress of DMU_o are more careful than the obtained results from $\left[MI_{1,p}^{o,L}, MI_{1,p}^{o,U}\right]$, because we consider all the time periods over two time periods t and $t+p-1$ computing $MI_{total}^{o,L}$ and $MI_{total}^{o,U}$, while they are neglected calculating $MI_{1,p}^{o,L}$ and $MI_{1,p}^{o,U}$.

4 Empirical example

In this section, in order to evaluate MIP, we consider 20 branch banks of Iran with 4 inputs and 5 outputs that their inputs and outputs lay within the bounded intervals. We are going to evaluate progress and regress of these branch banks over 5 months by our proposed method. The set of inputs and outputs are shown in Table 1. Note that, the interval data of inputs and outputs have not been shown for the sake of their voluminous. The evaluation results are shown in Table 2.

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Table 1 The set of inputs and outputs

Inputs		Outputs	
(I_1)	Number of year of establishment	(O_1)	Savings
(I_2)	Area	(O_2)	Deposits
(I_3)	Privilege of staff	(O_3)	Current account
(I_4)	Equipment	(O_4)	Invest for long time
		(O_5)	Invest for short time

Table 2 The evaluation results

Branch	$\left[MI_{total}^{L}, MI_{total}^{U}\right]$	$\left[MI_{1,5}^{L}, MI_{1,5}^{U}\right]$
1	[0.2628, 21.3570]	[1.5706, 3.5932]
2	[0.2628, 21.3570]	[0.5113, 2.2520]
3	[0.0188, 44.7686]	[0.4209, 1.5111]
4	[0.0012, 1932.5738]	[0.4363, 4.9744]
5	[0.0081, 329.5365]	[0.4160, 6.5533]
6	[0.0026, 913.5370]	[0.4159, 5.0791]
7	[0.1607, 19.1774]	[1.1635, 2.5233]
8	[0.2232, 14.2024]	[0.9353, 3.2530]
9	[0.4109, 5.9373]	[1.1433, 2.0861]
10	[0.0165, 1124.1851]	[1.4258, 14.0668]
11	[0.0498, 92.7945]	[0.8459, 3.7994]
12	[0.0274, 117.1401]	[0.7961, 3.9927]
13	[0.0170, 198.6825]	[0.5096, 6.9084]
14	[0.0003, 7744.6122]	[0.1880, 11.0214]
15	[0, 137011.3262]	[0.5030, 3.8440]
16	[0.5342, 4.3181]	[1.1095, 2.0335]
17	[0.0434, 52.0556]	[0.6732, 3.2822]
18	[0.0130, 191.4018]	[0.4820, 4.7686]
19	[0.0322, 120.9594]	[0.6954, 4.7947]
20	[0.0158, 230.5551]	[0.5634, 5.9375]

According to Table 2, $\left[MI_{total}^{L}, MI_{total}^{U}\right]$ indicates more percent of the progress with respect to regress for DMU_1, DMU_7, DMU_9, DMU_{10}, and DMU_{16}, while $\left[MI_{1,5}^{L}, MI_{1,5}^{U}\right]$ perfectly indicates progress for them and also, $\left[MI_{total}^{8,L}, MI_{total}^{8,U}\right]$ indicates more percent of the regress and less percent of the progress for DMU_8. Furthermore, $\left[MI_{total}^{L}, MI_{total}^{U}\right]$ indicates more percent of the progress and less percent of the regress for the rest of DMUs.

In this case study, the CCR DEA model (in an input-orientation) [14] have been used to compute the efficiency of branch banks in different months. It is obvious that the other DEA models can be used, too.

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5 Conclusions and future extensions

In this paper, we explore a method to compute the Malmquist productivity index for evaluating productivity change of a DMU in several time periods (from the first to the last periods) on interval data. Then it is compared with the Malmquist productivity index over two time periods (the first and last time periods) on interval data for assessing progress and regress of the target DMU. In section 2, we briefly describe to compute the Malmquist productivity index over two time periods on interval data for evaluating productivity change of a DMU. Our proposed method is presented in Section 3.

It seems that the obtained results from the Malmquist index in several time periods for evaluating progress and regress of the DMU under evaluation are more careful than the obtained results from the Malmquist index over two time periods, because all the time periods between the first and the last time periods are considered in our proposed method, while they are neglected in the previous methods.

At last, to reveal the proposed approach, we apply it to compute the Malmquist productivity index of bank branches to assess progress and regress of the DMU under evaluation. We suggest considering special data such as stochastic, interval, integer, fuzzy, etc. for future researches.

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