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Creating Full Envelopment in Data Envelopment Analysis with Variable Returns to Scale Technology

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Abstract In this paper, weak defining hyperplanes and the anchor points in DEA, as an important subset of the set of extreme efficient points of the Production Possibility Set (PPS), are used to construct unobserved DMUs and in the long run to improve the envelopment of all observed DMUs. There has been a surge of articles on improving envelopment in recent years. What has been done first is in Constant Returns to Scale (CRS) environment for single input multiple output cases, and the latter study is in Variable Returns to Scale (VRS) environment for multiple input and output cases, but none of them guarantees full envelopment. We provide a short summary of what has been done in VRS environments to improve envelopment; we then devise an algorithm to create full envelopment of all observed DMUs under VRS technology. We illustrate our algorithm using a numerical example.

Keywords: Data Envelopment Analysis, Anchor Points, Unobserved DMUs, Weak Defining Hyperplanes, Efficiency.

1 Introduction

Data Envelopment Analysis (DEA), introduced by Farrel [1] and Charnes, et al., [2], is linear programming based method used to evaluate relative efficiency of a Decision Making Unit (DMU) from a set of n DMUs. Based on various production process assumptions, a number of different models have been developed. The original DEA model by Charnes et al., [2] which assumes Constant Returns to Scale (CRS) was extended by Banker et al., [3] to allow for Variable Returns to Scale (VRS) technologies. Under VRS an output efficiency score yielded by DEA for a DMU is based on the ratio of the sum of its weighted outputs plus a variable for the DMU's scale of operation to the sum of its weighted input levels. The weights are free to be estimated in model in order to maximize the efficiency score of the evaluated DMU, only subject to the constrains that all weights should be greater than a minimum value of ε, a very small number (non-Archimedian infinitesimal).

There is a lot of reasons that recognize this near complete weight flexibility in DEA often leads to unacceptable results. One reason is that not all the inputs and (or) outputs are given sufficient weight in computing the efficiency scores. Clearly, one straightforward way to overcome this problem is to raise the lower bound ε on the DEA-weights to a level that is

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deemed sufficient (see e.g.; Dyson and Thanassoulis [4], Allen et al. [5] and Thanassoulis et al. [6]). In the VRS environment weight restricted DEA models lead frequently to inappropriate results such as negative efficiency scores and infeasibilities as demonstrated in Allen [7] and Estellita Lins et al. [8]. The next study also suggests a new approach for weight restrictions that avoids infeasibilities under VRS. However, the model by Estellita Lins et al. [8] is nonlinear and does not produce radial efficiency scores. Obviously, these are relevant limitations for this approach, because many non-radial models can also avoid unreasonable weights to inputs and (or) outputs. Thus, there is a need for an alternative approach for capturing value judgments in a VRS environment, which does not require nonlinear models or non-radially.

The latter study by Thanassoulis et al. [9] recommends an approach that utilizes the envelopment DEA model in including additional technical and value information in efficiency evaluation. In order to improve envelopment the approach incorporates in the assessment the Decision Maker's (DM's) values or other information about the input output transformation process by explicitly modifying the Production Possibility Set (PPS); (see Banker et al. [3]). Therefore, this approach perceives the concept of the inclusion of values (value judgments) in DEA as that of missing data. In other words, efficient levels of inputs and outputs for operating processes which at present are only observed at inefficient levels are to be estimated. The approach is only attempting to extend the observed Pareto-efficient frontier, rather than alter and extend the observed frontier.

The approach developed essentially takes forward the ideas in Thanassoulis and Allen [10], who demonstrated the equivalence between relative DEA weight restrictions and the incorporation of Unobserved DMUs (UDMUs). Those ideas were used in Allen and Thanassoulis [11] to create UDMUs and thereby incorporate values in the DEA assessment process. However, Allen and Thanassoulis [11] covered the limited single input multiple output CRS case. This approach generalizes the method to the multiple input and output VRS environment. In the multi-input multi-output case the choice of inputs and (or) output to modify and the direction of modification in order to create UDMUs is considerably more complex than in the single input multi-output case. Further, the VRS technology makes it necessary to reflect the nature of returns to scale when creating UDMUs. Thanassoulis et al. [9] address these issues and provide an appropriate heuristic for improving envelopment in this more complex setting. The approach has some similarities with the constrained facet approaches (see e.g.; Bessent et al. [12], Lang et al. [13], Olesen and Petersen [14]), which also operate directly on the PPS instead of weight restrictions. However, the difference from constrained facet approaches is that, this approach uses the DM's preferences to extend the PPS. The main idea behind the approach is to extend the Pareto-efficient frontier by adding suitably constructed UDMUs in the data set.

This offers a number of benefits over weights restrictions as far as incorporating values in a DEA assessment is concerned. The main advantages and disadvantages of UDMUs in comparison to weight restrictions are detailed in Thanassoulis and Allen [10], Allen and Thanassoulis [11] and Thanassoulis et al. [9]. Other studies dealing with unobserved DMUs include Sowlati and Paradi [15], Jahanshahloo and Soleimani-Damaneh [16] and Diallo et al. [17]. Awhile back, UDMUs have been utilized in generalizing weak disposable DEA technologies (see Kuosmanen [18], Kuosmanen and Podinovski [19], Podinovski and Kuosmanen [20]). However, this approach does not use DM preferences in incorporating DMUs, but applies the weak disposable property to construct UDMUs under VRS. Incidentally, the proposed approach by Thanassoulis et al. [9] even though extends the Pareto-efficient frontier and develops envelopment, does not guarantee full envelopment of all

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DMUs; and it means that after applying the proposed approach, not covered DMUs may still exist. To overcome this defect, in this paper we propose another new approach which restricts DM's liberty and guarantees full envelopment of all DMUs by creating some changes in the proposed approach by Thanassoulis et al. [9]).

The rest of the paper is organized as follows. Section 2 provides a summary of what Thanassoulis et al. [9] have done to improve envelopment. Section 3 and 4 detail the steps which determine UDMUs and provide full envelopment. Section 5 presents an application to a set of data to illustrate the approach. Finally the paper concludes and suggests further developments of the approach.

2 A summary of improving envelopment by creating UDMUs

The proposed general procedure for improving envelopment under multiple inputs and outputs in a VRS technology by Thanassoulis et al. [9] can be summarized as follows:

I. Assess the DMUs in the appropriate orientation to determine the Pareto efficient DMUs and the envelopment of the inefficient DMUs. If there are non-enveloped DMUs go to (II), otherwise stop.
II. Identify the Anchor DMUs (ADMUs), which are the (extreme) efficient DMUs from where the Pareto efficient frontier is to be extended.
III. For each ADMU, identify which input and (or) output levels of the DMUs need to be individually raised or lowered to improve envelopment.
IV. Construct estimates of Pareto efficient UDMUs by reference to the DM.
V. Re-assess the DMUs permitting both DMUs and UDMUs to define the efficient boundary.
VI. If all DMUs are fully enveloped or the DM feels a sufficient number of DMUs are fully enveloped stop. Otherwise repeat steps (IV) and (V).

The steps above can increase the number of properly enveloped DMUs in the assessment, but do not guarantee full envelopment.

3 The proposed approach

Our new proposed initiative which guarantees full envelopment under multiple inputs and outputs in a VRS technology is as follows:

i. Assess the DMUs in the appropriate orientation to determine the Pareto efficient DMUs and the envelopment of the inefficient DMUs. If there are non-enveloped DMUs go to (ii), otherwise stop.
ii. Identify the Anchor DMUs (ADMUs), which are the (extreme) efficient DMUs from where the Pareto efficient frontier is to be extended.
iii. Find the average point on weak hyperplanes by obtaining all of the weak hyperplanes.
iv. Find a point out of T_V by reducing input of the average point for the input orientation or increasing output of the average point for the output orientation.
v. Determine the lowest level of outputs for the input orientation case or the highest level of inputs for the output orientation case between inefficient DMUs.

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vi. Adjust the obtained point in step (iv) by using step (v) to create UDMUs.
vii. Re-evaluate DMUs in the presence of the created UDMUs to see all DMUs are enveloped and full envelopment is achieved.

The sections which follow explain each one of the steps (i) – (vii) above.

Consider the set of N DMUs with m inputs x_{ij}, i=1,…,m to produce s different outputs y_{rj}, r=1,…,s under variable returns to scale and in the input orientation (though a similar procedure can be developed for the output orientation Case). The following steps will ensure full envelopment of DMUs under multiple inputs and outputs in a VRS technology.

3.1 Step (i): Primary assessing of DMUs

We will utilize here the classifications of DMUs in DEA introduced by Charnes et al. [21]. For ease of reference, they are summarized here:

- Class E^*: Extreme efficient DMUs which cannot be expressed using linear combinations of other extreme efficient DMUs, determined by the variable returns to scale assumptions of the model.
- Class E: Pareto efficient DMUs which are not class E^*.
- Class F: Inefficient DMUs that are on the PPS boundary.
- Class NE^*, NE or NF: DEA inefficient DMUs such that their radial projection on the PPS boundary constructs a class E^*, E or F DMU, respectively. It is important to note that in the general VRS environment this classification based on projections is affected by the orientation used.

By using model (M1), identify the set of Pareto efficient DMUs which are categorized in class E^* and E, weak efficient DMUs which are categorized in class F and also inefficient DMUs which are categorized in class NE^*, NE and NF. Indeed by solving model (M1), Pareto and weak efficient frontiers are distinguished.

$$h_{j0} = Min\ f_0 - \varepsilon \left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r} \right)$$

$$s.t.$$

$$\begin{aligned} & f_0 x_{ij_0} - \sum_{j=1}^{N} \lambda_j x_{ij} - S_i = 0 \qquad i = 1,\ldots,m \\ & \sum_{j=1}^{N} \lambda_j y_{rj} - S_{m+r} = y_{rj_0} \qquad r = 1,\ldots,s \\ & \sum_{j=1}^{N} \lambda_j = 1 \\ & \lambda_j, S_i, S_{m+r} \geq 0 \quad \forall j, i, r. \end{aligned} \tag{M1}$$

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If all DMUs are enveloped stop, otherwise go to the next step.

3.2 Step (ii): Identifying the anchor DMUs (ADMUs)

Let JE^* be the set of extreme-efficient DMUs. To identify whether DMU is ADMU, solve model (M2):

$$h_{j0} = Min\ f_0 - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$$

$$s.t.$$

$$\begin{aligned}
&\sum_{j \in JE^*_{j0}} \lambda_j x_{ij} + S_i - f_0 x_{ij_0} = 0 \qquad i = 1,\ldots,m \\
&\sum_{j \in JE^*_{j0}} \lambda_j y_{rj} - S_{m+r} = y_{rj_0} \qquad r = 1,\ldots,s \\
&\sum_{j \in JE^*_{j0}} \lambda_j = 1 \\
&\lambda_j, S_i, S_{m+r} \geq 0 \quad \forall j \in JE^*_{j0}, \forall i, r
\end{aligned} \tag{M2}$$

Where x_{ij} and y_{rj} are as defined in (M1). DMU_{j0} must meet either one of the following conditions to be classed as an ADMU:

- $h_{j0} > 1$ and at least one $S_i > 0$ or $S_{m+r} > 0$ in (M2) or
- (M2) has no feasible solution.

3.3 Step (iii): Finding the average point on weak hyperplanes

Obtain all of the weak hyperplanes. Suppose that H^i is a weak hyperplane; put:

$$A^i = \{\text{Indexes of all anchor units on the hyperplane } H^i\} \tag{1}$$

Calculate:

$$\begin{pmatrix} \overline{X}^i \\ \overline{Y}^i \end{pmatrix} = \sum_{i \in A^i} \frac{1}{|A^i|} \begin{pmatrix} X_i \\ Y_i \end{pmatrix} \tag{2}$$

Actually, we obtain the point that is the average point on the weak hyperplane. In Theorem 1, we prove that the obtained point is on the weak hyperplane H^i.

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Theorem 3.1. The point $\begin{pmatrix} \overline{X}^i \\ \overline{Y}^i \end{pmatrix}$ is on the weak hyperplane H^i.

See Appendix for proof.

3.4 Step (iv): Finding a point out of T_V

The purpose is to find a point out of T_V, so that the weak hyperplanes can be removed to assess DMUs. For this, the new point is defined as follow:
Put:

$$\begin{pmatrix} \overline{X}^i \\ \overline{Y}^i \end{pmatrix} = \begin{pmatrix} \overline{X}^i - \varepsilon 1_m \\ \overline{Y}^i \end{pmatrix} \tag{3}$$

Note that the point $\begin{pmatrix} \overline{X}^i \\ \overline{Y}^i \end{pmatrix}$ is a point on the weak hyperplane (weak frontier) which is parallel to the output axis; So it is expected that by reducing input, the weak hyperplane is not in T_V.

In Theorem 2, it will be shown that the point $\begin{pmatrix} \overline{X}^i - \varepsilon 1_m \\ \overline{Y}^i \end{pmatrix}$ is out of T_V.

Theorem 3.2. Point $\begin{pmatrix} \overline{X}^i - \varepsilon 1_m \\ \overline{Y}^i \end{pmatrix}$ is out of the weak hyperplane H^i, therefore is out of T_V.

See Appendix for proof.

3.5 Step (v): Determining the lowest level of outputs

Find the lowest level of similar components of outputs between inefficient DMUs belong to class NF that are distinguished by (M1) in the First step, and name the obtained output vector Y^{min}.

$$\forall\, j;\ DMU_j \in \text{class NF}\ ; \quad Y^{min} = \begin{pmatrix} y_1 = \min\{y_{1j}; \forall j\} \\ y_2 = \min\{y_{2j}; \forall j\} \\ \vdots \\ y_s = \min\{y_{sj}; \forall j\} \end{pmatrix} \tag{4}$$

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3.6 Step (vi): Creating UDMUs

At the point $\begin{pmatrix} \overline{X}^i - \varepsilon 1_m \\ \overline{Y}^i \end{pmatrix}$ instead of $\overline{Y}^i$, put Y^{min} as follow:

$$\begin{pmatrix} \overline{X}^i - \varepsilon 1_m \\ \overline{Y}^i \end{pmatrix} = \begin{pmatrix} \overline{X}^i - \varepsilon 1_m \\ Y^{min} \end{pmatrix} \quad (5)$$

The obtained point above is a UDMU that by using this, we want to extend the Pareto efficient frontier and create full envelopment.

3.7 step (vii): Re-evaluating DMUs in the presence of the created UDMUs

As described at third until sixth steps, create a UDMU for each weak hyperplane which contains the anchor DMU. Add all created UDMUs to the initial DMUs set, and re-create the new set T_v. Re-assess the initial DMUs plus the created UDMUs (new T_v) by solving Model (M1); and you will see that all DMUs are enveloped and full envelopment is achieved.

Note that the determination of ε value is up to the DM. So the DM should choose a value for ε, which the created UDMUs do not destroy the Pareto efficiency of anchor DMUs; and this means that the returns to scale of created UDMUs has to be consistent with the returns to scale of anchor DMUs. For this purpose, it is enough $\varepsilon \to 0^+$.

A summary of the foregoing steps is now given, in the form of a heuristic.

4 A procedure to guarantee full envelopment in DEA via UDMUs under VRS

Consider a set of N DMUs using m inputs, x_{ij}, i=1,…,m to produce s different outputs y_{rj}, r=1,…,s, where an assumption of variable returns to scale is maintained and the input orientation is appropriate for assessing efficiency. The following steps can guarantee full envelopment of all DMUs in the assessment.

i. Apply model (M1) to identify the set of Pareto efficient DMUs which are of class E^* and E, weak efficient DMUs which are of class F and also inefficient DMUs which are of class NE^*, NE and NF as defined by Charnes et al. [21]. If all DMUs are properly enveloped stop. Otherwise go to (ii).

ii. For each $j \in JE^*$ (i.e. for extreme efficient units) solve model (M2) to determine h_{j0} as defined in that model. Hence identify the set of ADMUs JA = {j | $h_{j0} > 1$, and at least one $S_i > 0$ or $S_{m+r} > 0$, or DMU_j has no feasible solution in (M2)}.

iii. Obtain all of the weak hyperplanes, for each weak hyperplane H^i by using (1) and (2) obtain its average point.

iv. For each average point on the weak hyperplanes obtained in step (iii), reduce the input vector by using (3) to find a point out of T_V.

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v. For all inefficient DMUs belong to class NF that are distinguished by (M1) in the step (i), find the output vector Y^{min} as shown in (4).

vi. For each point out of T_V obtained in step (iv), replace its output vector with output vector Y^{min} determined in step (v) as indicated in (5) to create UDMUs.

vii. Assess the DMUs using model (M1) but permitting both DMUs and the UDMUs created in step (vi) to define the PPS boundary. All DMUs would be properly enveloped and full envelopment would be achieved; so stop.

The foregoing heuristic can be readily modified for an output maximization model. The next section demonstrates the use of the foregoing process on a real data set.

5 Illustration of the use of UDMUs to create full envelopment in DEA

In this section the use of UDMUs to create and guarantee full envelopment will be illustrated by applying the procedure developed to a set of 9 DMUs with the input output variables of Table 1.

Table 1 Data of inputs and outputs of DMUs

DMUs	D1	D2	D3	D4	D5	D6	D7	D8	D9
x_1	26	20	22	15	30	35	34	20	22
x_2	20	15	10	12	22	25	24	16	17
y_1	15.5	17.2	14.3	14	12	12	12	14.3	13.5
y_2	8	5	8	5	3	2	2	10	8

5.1 Step (i): The initial evaluation of the DMUs

The initial step of the procedure is to evaluate the DMUs using model (M1) to identify the Pareto efficient DMUs and establish whether there is a need to improve the envelopment of the DMUs. It was found that DMUs D1, D2, D3, D4 and D8 are of class E^*, and DMUs D5, D6, D7 and D9 are of class NF. This result establishes a need for a procedure to improve the envelopment of the inefficient DMUs.

5.2 Step (ii): The anchor DMUs of the evaluation

Using model (M2) it was found that 2 of the 5 Pareto efficient DMUs were anchor DMUs. It means that DMUs D3 and D4 are anchor DMUs.

5.3 Step (iii): The average point on weak hyperplanes

All weak hyperplanes were found by using the proposed algorithm by Davtalab-Olyaie et al. [22]. For weak hyperplane $H^1: 0.11y_1 - 0.06x_2 - 0.97 = 0$ and the anchor DMU D3 lying on it,

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the average point $\begin{pmatrix}\overline{X}^1\\ \overline{Y}^1\end{pmatrix}=\begin{pmatrix}22,10\\ 14.3,8\end{pmatrix}$, for weak hyperplane $H^2:0.06y_1-0.01x_1-0.04x_2-0.16=0$ and the anchor DMUs D3 and D4 lying on it, the average point $\begin{pmatrix}\overline{X}^2\\ \overline{Y}^2\end{pmatrix}=\begin{pmatrix}18.5,11\\ 14.15,6.5\end{pmatrix}$ and for weak hyperplane $H^3:0.07y_1+0.04y_2-0.05x_1-0.57=0$ and the anchor DMU D4 lying on it, the average point $\begin{pmatrix}\overline{X}^3\\ \overline{Y}^3\end{pmatrix}=\begin{pmatrix}15,12\\ 14,5\end{pmatrix}$ were obtained.

5.4 Step (iv): The points out of T_V

The input vectors of all average points obtained in step (iii) were reduced and the points $\begin{pmatrix}\overline{X}^1-\varepsilon 1_m\\ \overline{Y}^1\end{pmatrix}=\begin{pmatrix}21.99,9.99\\ 14.3,8\end{pmatrix}$, $\begin{pmatrix}\overline{X}^2-\varepsilon 1_m\\ \overline{Y}^2\end{pmatrix}=\begin{pmatrix}18.49,10.99\\ 14.15,6.5\end{pmatrix}$ and $\begin{pmatrix}\overline{X}^3-\varepsilon 1_m\\ \overline{Y}^3\end{pmatrix}=\begin{pmatrix}14.99,11.99\\ 14,5\end{pmatrix}$ which are out of T_V, were obtained.

5.5 Step (v): The lowest level of outputs

For all inefficient DMUs belong to class NF, the output vector $Y^{min}=\begin{pmatrix}12\\ 2\end{pmatrix}$ was found.

5.6 Step (vi): The created UDMUs

By replacing output vectors of points out of T_V obtained in step (iv), with output vector $Y^{min}=\begin{pmatrix}12\\ 2\end{pmatrix}$, the UDMUs $\begin{pmatrix}\overline{X}^1-\varepsilon 1_m\\ \overline{Y}^{min}\end{pmatrix}=\begin{pmatrix}21.99,9.99\\ 12,2\end{pmatrix}$, $\begin{pmatrix}\overline{X}^2-\varepsilon 1_m\\ \overline{Y}^{min}\end{pmatrix}=\begin{pmatrix}18.49,10.99\\ 12,2\end{pmatrix}$ and $\begin{pmatrix}\overline{X}^3-\varepsilon 1_m\\ \overline{Y}^{min}\end{pmatrix}=\begin{pmatrix}14.99,11.99\\ 12,2\end{pmatrix}$ were created.

5.7 Step (vii): Evaluation of the DMUs permitting UDMUs to be on the boundary

Finally, the DMUs with the inclusion of the created UDMUs were assessed again in the context of model (M1). This was found that all DMUs are fully enveloped and full envelopment was achieved.

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6 Conclusion

This paper has extended the approach developed in Thanassoulis et al. [9] from introducing values and improving envelopment in a VRS DEA assessment with multiple inputs and outputs to a new approach while also guaranteeing full envelopment. The approach utilizes UDMUs created by using certain observed DMUs, identified as 'Anchor' DMUs and weak defining hyperplanes.

Creating UDMUs by finding the average points of ADMUs on weak defining hyperplanes offers a significant advantage over creating UDMUs by adjusting the input–output levels of ADMUs; it is guaranteed that applying the proposed approach in this paper once, makes full envelopment of all observed DMUs. Moreover, it reduces the number of created UDMUs and this means that the amount and complexity of the calculations will be less.

Evidently the difficulty in the approach is to find weak hyperplanes which requires to solve two models several times and takes time, however may exist easier methods to find weak hyperplanes. Although the one presented here will guarantee full envelopment of all DMUs, there may be alternative approaches and extending the method to a new approach which is less time consuming is an area for further research.

Appendix 1.

Proof of Theorem 3.1. Suppose that $A_1,...,A_t \in H^i$, and

$$H^i: U^tY - V^tX - u_0 = 0$$

If $A_1 = \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \in H^i$, so

$$U^tY_1 - V^tX_1 = u_0$$

If $A_2 = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \in H^i$, so

$$U^tY_2 - V^tX_2 = u_0$$

and similarly if $A_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix} \in H^i$, so

$$U^tY_t - V^tX_t = u_0$$

From sum of the above equations:

$$U^t(Y_1+Y_2+...+Y_t) - V^t(X_1+X_2+...+X_t) = u_0+...+u_0$$

The number of u_0s are t. Both sides divided by t makes:

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$$U^t\left(\frac{Y_1+...+Y_t}{t}\right)-V^t\left(\frac{X_1+...+X_t}{t}\right)=u_0$$

as a result, we will have:

$$\begin{pmatrix}\frac{X_1+...+X_t}{t}\\ \frac{Y_1+...+Y_t}{t}\end{pmatrix}\in H^i$$

And it means:

$$\begin{pmatrix}\overline{X}^i\\ \overline{Y}^i\end{pmatrix}\in H^i \quad \square$$

Proof of Theorem 3.2. We have:

$$U^i\overline{Y}^i-V^i\overline{X}^i=u_0^i$$

Put:

$$\overline{X}^i=\overline{X}^i-\varepsilon 1_m$$

Will have:

$$U^i\overline{Y}^i-V^i(\overline{X}^i-\varepsilon 1_m)=$$

$$U^i\overline{Y}^i-V^i\overline{X}^i+\varepsilon V^i 1_m=$$

$$u_0^i+\varepsilon V^i 1_m>u_0^i$$

So point $\begin{pmatrix}\overline{X}^i-\varepsilon 1_m\\ \overline{Y}^i\end{pmatrix}$ is out of the weak hyperplane H^i and so is out of T_V. $\square$

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