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# Global Optimization of Fractional Posynomial Geometric Programming Problems Under Fuzziness

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**Abstract** In this paper we consider a global optimization approach for solving fuzzy fractional posynomial geometric programming problems. The problem of concern involves positive trapezoidal fuzzy numbers in the objective function. For obtaining an optimal solution, Dinkelbach's algorithm which achieves the optimal solution of the optimization problem by means of solving a sequence of subproblems is extended to the proposed problem. In addition, An illustrative example is included to demonstrate the correctness of the proposed solution algorithm.

**Keywords:** Trapezoidal Fuzzy Nnumber, Fractional Programming, Global Optimization, Posynomial Geometric Programming, Parametric Approach.

## 1 Introduction

In most of the practical situations the possible value of the parameters involved in objective could not be defined precisely due to the lack of available data. The concept of fuzzy sets [1] is seemed to be most appropriate to deal with such imprecise data. To deal with fuzziness, fuzzy programming have been proposed to make decisions under an uncertainty environment. Cao was the pioneer in the research topic of fuzzy geometric programming problem [2, 3].

The optimization problem in which the objective function appears as a ratio of two functions is known as a fractional programming (FP) problem [4-6]. Fractional objectives appear in many real world situations. For instance, we often need to optimize the efficiency of some activities like cost/time, cost/profit, and output/employee. For an overview of these applications, we refer to [7, 8] and the references therein. The optimization problem involving imprecise parameters in FP are called fuzzy fractional programming (FFP) problem [9-12].

There are different solution algorithms for determining the optimal solution of particular kinds of fractional programming problems. Charnes and Cooper [13] proposed an exact linear programming reformulation of the continuous linear fractional program. One of the popular solution methods was first introduced by Martos and Andrew Whinston [14] and Jagannathan

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[15] and they named it as the so-called "parametric" approach. The main idea of this parametric approach is to solve an equivalent parametric problem of the fractional program. Dinkelbach [16] extended the parametric approach to solve the continuous nonlinear fractional programs using the Newton's method. It is worth mentioning that Dinkelbach's method is valid for fractional problems with objectives being minimized or maximized. Several authors extended Dinkelbach's approach to solve several problems involving fractional objectives such as generalized fractional programming problems [6, 17] and the minimum spanning tree with sum of ratios problems [18]. Almogy and Levin [19] extended the parametric approach of Dinkelbach to solve sum of ratios problems. Tammer et al. [20] and Valipour et al. [21] extended Dinkelbach's approach to solve multiobjective linear fractional programming (MOLFP) problems. An algorithm based on the parametric approach was proposed to solve integer linear fractional programming problems by Ishii et al. [22]. Pochet and Warichet [23] and You et al. [24] showed that the parametric approach is very efficient for solving mixed-integer linear fractional programming (MILFP) models for cyclic scheduling. Yue, Guillén-Gosálbez and You [25] proposed an exact mixed-integer linear programming (MILP) reformulation for large-scale MILFP problems, although it cannot be applied to mixed-integer non-linear fractional programs. Zhong and You [26] is concerned with the parametric algorithms for solving large-scale mixed-integer linear and nonlinear fractional programming problems. Chu and You [27] developed an efficient global optimization algorithm for the MINLFP master problem that is based on a parametric fractional programming approach. Also to optimize the non convex MINLFP problems, Gong et al. [28] and Chu and You [29] proposed the global optimization strategies based on the Dinkelbach's algorithm. Although Dinkelbach's approach has been used to solve many different problems involving fractional objectives, there is no absolutely successful extension to solve fuzzy fractional posynomial geometric programming (FFPGP) problems.

The current paper attempts to propose an iterative algorithm that extends Dinkelbach's approach to solve a fractional posynomial geometric programming problem with positive trapezoidal fuzzy coefficients in objective function. This paper is organized as follows: fuzzy notations and definitions used in the remaining parts of the paper are presented in Section 2. Section 3 contains the mathematical formulation of fuzzy fractional posynomial geometric programming problem and its solving procedure. In addition, parameterized form of the problem concern is described by proving some theorems. An illustrative example is given in Section 4 to clarify the solution algorithm. The paper ends with conclusions in Section 5.

## 2 Preliminaries

In this section, we give some notions and definitions on which our research in this paper is based.

Fuzzy sets first introduced by Zadeh [1] as a mathematical way of representing vagueness in everyday life. According to [30], The characteristic function $\mu_A$ of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in $X$. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set $X$ fall within a specified range i.e. $\mu_{\tilde{A}} : X \to [0,1]$. The assigned value indicates the *membership grade* of the element in the set $A$. The function $\mu_{\tilde{A}}$ is called the *membership function* and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$ defined by $\mu_{\tilde{A}}(x)$ for each $x \in X$ is called a *fuzzy set*. A fuzzy set

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$\tilde{A}$, defined on the universal set of real numbers $\mathrm{R}$, is said to be a *fuzzy number* if its membership function has the following characteristics:

1. $\mu_{\tilde{A}} : \mathrm{R} \to [0,1]$ is continuous.
2. $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a] \cup [d, \infty)$.
3. $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a,b]$ and strictly decreasing on $[c,d]$.
4. $\mu_{\tilde{A}}(x) = 1$ for all $x \in [b,c]$, where $a < b < c < d$.

**Definition 2.1** [30] A fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be a trapezoidal fuzzy number if it's membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & a < x < b, \\ 1, & b \le x \le c, \\ \dfrac{(x-d)}{(c-d)}, & c < x < d. \end{cases} \tag{1}$$

The $\alpha$-cut (level set) [31] of fuzzy number $\tilde{A}$ can be obtained as: $(\tilde{A})_\alpha = A_\alpha = \{x \mid \mu_{\tilde{A}}(x) \ge \alpha\}$, for example, let $\tilde{A}$ be a trapezoidal fuzzy number, to find the $\alpha$-cut of $\tilde{A}$, we first set $\alpha \in [0,1]$ to both left and right reference functions of $\tilde{A}$, that is, $\alpha = \dfrac{(x-a)}{(b-a)}$ and $\alpha = \dfrac{(x-d)}{(c-d)}$. Expressing $x$ in terms of $\alpha$, we have $x = (b-a)\alpha + a$ and $x = -(d-c)\alpha + d$ which gives the $\alpha$-cut of $\tilde{A}$ as:

$\tilde{A}_\alpha = [A^-(\alpha), A^+(\alpha)] = [(b-a)\alpha + a, -(d-c)\alpha + d]$.

**Definition 2.2** [32] A trapezoidal fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be positive (negative) trapezoidal fuzzy number, denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$), if and only if $a > 0$ ($c < 0$).

**Definition 2.3** [33] Let $a > 0, b > 0$ and consider the interval $[a,b]$. From a mathematical point of view, any real number can be represented on a line. Similarly, we can represent an interval by a function. If the interval is of the form $[a,b]$, the interval function is taken as $h(q) = a^{1-q} b^q$, *for* $q \in [0,1]$.

According to [34], a *signomial function* is defined as the sum of signomial terms, which in turn consists of products of power functions. Thus, a signomial function can be expressed mathematically as $\sigma(x) = \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}$, where the coefficients $c_k$ and the powers $\gamma_{kl}$ are real. A special type of signomial function, where all coefficients $c_k > 0, k = 1,...,J$, is called *posynomial function*.

**Definition 2.4** [35] A posynomial geometric programming (PGP) problem can be stated as:

Find $x = (x_1, x_2, ..., x_m)^T$ so as to

$$\begin{aligned} &\min \quad g_0(x) = \sum_{k=1}^{J_0} c_{0k} \prod_{l=1}^{m} x_l^{\gamma_{0kl}}, \\ &s.t. \\ &\quad g_i(x) = \sum_{k=1}^{J_i} c_{ik} \prod_{l=1}^{m} x_l^{\gamma_{ikl}} \le 1, \quad i = 1,2,...,p, \\ &\quad x_l > 0, \quad l = 1,2,...,m, \end{aligned} \tag{2}$$

where $c_{0k}$ and $c_{ik}$ are positive real constant coefficients for all $i,k$;

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$\gamma_{0kl}$ and $\gamma_{ikl}$ are arbitrary real constant exponents for all $i,k,l$;
$J_0$ is the number of terms present in the objective function $g_0(x)$;
$J_i$ is the number of terms present in the $i$-th constraint, $i=1,2,...,p$.

## 3 Problem formulation and solution concept

In this section, a fuzzy fractional posynomial geometric programming problem and its global optimization based on the iterative parametric approach are described.

### 3.1 Problem formulation

The problem to be considered in this paper is the following fuzzy fractional posynomial geometric programming (FFPGP) problem:
Find $x=(x_1,x_2,...,x_m)^T$ so as to

$$\max \quad \frac{\tilde{g}_1(x)=\sum_{k=1}^{J_1}\tilde{c}_{1k}\prod_{l=1}^{m}x_l^{\gamma_{1kl}}}{g_2(x)=\sum_{k=1}^{J_2}c_{2k}\prod_{l=1}^{m}x_l^{\gamma_{2kl}}}$$
$$s.t. \tag{3}$$
$$x\in X=\{x;0<x^L\le x\le x^U\},$$

where $x=(x_1,x_2,...,x_m)^T$ is a variable vector, and $T$ stands for transpose;
the feasible region $X$ is nonempty, compact and bounded;
$c_{2k}$ are positive real constant coefficients for all $k$;
$\gamma_{1kl}$ and $\gamma_{2kl}$ are arbitrary real constant exponents for all $k,l$;
$\tilde{c}_{1k}=(a_{1k},b_{1k},c_{1k},d_{1k})$ are positive trapezoidal fuzzy numbers;
$J_1$ and $J_2$ represent the number of product terms of numerator and of denominator in the objective function, respectively;
$\tilde{g}_1(x)$ and $g_2(x)$ are fuzzy posynomial function and posynomial function, respectively, and $g_2(x)$ is positive for all $x$ in the feasible region $X$.

### 3.2 solution concept

In this subsection, a global optimization approach for solving the problem (3) involving positive trapezoidal fuzzy coefficient in objective function is presented.

#### 3.2.1 Formulation based on interval function

At first for a certain degree $\alpha=\alpha^*\in[0,1]$, estimated by the decision maker, the problem (3) can be understood as the following nonfuzzy $\alpha$-fractional posynomial geometric programming ($\alpha$-FPGP) problem:

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$$Max \quad \frac{\sum_{k=1}^{J_1}[c_{1k}^{-}(\alpha), c_{1k}^{+}(\alpha)]\prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

s.t. (4)

$$x \in X = \{x; 0 < x^L \le x \le x^U\}.$$

Next based on Definition 2.3 in section 2, optimization problem (4) can be written as the following equivalent problem $(P_q)$:

$$Max \quad \frac{\sum_{k=1}^{J_1}(c_{1k}^{-}(\alpha))^{1-q}(c_{1k}^{+}(\alpha))^{q}\prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

*s.t.* (5)

$$x \in X = \{x; 0 < x^L \le x \le x^U\},$$
$$0 \le q \le 1.$$

The following theorem gives the idea that the $\alpha$-optimal solution of the problem (4) is possible to find in the form of problem (5),(see [33]).

**Theorem 3.1** the problem (5) provides the solution of the problem (4).

**Proof**. For any $k$, if we take $\beta_k \in [c_{1k}^{-}(\alpha), c_{1k}^{+}(\alpha)]$, the problem (4) reduces to

$$Max \quad \frac{\sum_{k=1}^{J_1}\beta_k \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

*s.t.* (6)

$$x \in X = \{x; 0 < x^L \le x \le x^U\}.$$

Let us consider the interval function $h(q) = a^{1-q}b^q$ for $q \in [0,1]$ and for an interval $\beta \in [a,b]$. Since $h(q)$ is a strictly monotone increasing and continuous function, the above problem reduces to:

$$Max \quad \frac{\sum_{k=1}^{J_1}\beta_k \prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{\sum_{k=1}^{J_2} c_{2k} \prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

*s.t.* (7)

$$x \in X = \{x; 0 < x^L \le x \le x^U\},$$

where $\beta_k \in (c_{1k}^{-}(\alpha))^{1-q}(c_{1k}^{+}(\alpha))^{q}$ and $q \in [0,1]$.

Since $h(q) = a^{1-q}b^q$ for $q \in [0,1]$ is a strictly monotone and continuous function, its inverse exists. Let $\delta$ be the inverse of $h(q)$, then $q = \frac{\log \delta - \log a}{\log b - \log a}$, therefore, we can find any particular $\beta$ for some values of $q \in [0,1]$.

Thus we can find the $\alpha$-optimal solution of the problem (4) only by solving the problem (5).

Note that for $q = 0$, the lower bound of the interval value of the parameter $q$ is used to find the optimal solution, so the following $(P_0)$ problem yields the lower bound of the optimal solution of problem (5).

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$$Max \quad \frac{g_{1\alpha}^{-}(x)=\sum_{k=1}^{J_1} c_{1k}^{-}(\alpha)\prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{g_2(x)=\sum_{k=1}^{J_2} c_{2k}\prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

*s.t.* (8)

$$x \in X=\{x;0<x^L \le x \le x^U\},$$

also, the case $q=1$ means that the upper bound of the interval parameter $q$ is used for finding the optimal solution, then the following $(P_1)$ problem yields the upper bound of the optimal solution of problem (5).

$$Max \quad \frac{g_{1\alpha}^{+}(x)=\sum_{k=1}^{J_1} c_{1k}^{+}(\alpha)\prod_{l=1}^{m} x_l^{\gamma_{1kl}}}{g_2(x)=\sum_{k=1}^{J_2} c_{2k}\prod_{l=1}^{m} x_l^{\gamma_{2kl}}}$$

*s.t.* (9)

$$x \in X=\{x;0<x^L \le x \le x^U\}.$$

Two optimization problems (8) and (9) can be more tractable by adopting Dinkledbach's parametric approach [16] as we will see in the following subsection.

### 3.2.2 Equivalent parametric problem and its properties

Consider the two fractional geometric programming problems (8) and (9) where their numerator and denominator are continuous posynomial functions. Using a parametric approach in [16], the above (8) and (9) problems can be solved indirectly by finding the solution to the following two equivalent parametric problems $(P_{\lambda^-})$ and $(P_{\lambda^+})$, respectively, i.e.,

$$Q(\lambda^-)=max\{g_{1\alpha}^{-}(x)-\lambda^- g_2(x);x\in X\}, \tag{10}$$

and

$$Q(\lambda^+)=max\{g_{1\alpha}^{+}(x)-\lambda^+ g_2(x);x\in X\}, \tag{11}$$

where $\lambda^-$(resp. $\lambda^+$) is a parameter. For a fixed parameter $\lambda^-$(resp. $\lambda^+$), the parametric problem (10) (resp. (11)) is typically easier to solve than the fractional geometric programming problem (8) (resp. (9)).

In what follows, since the properties of function $Q(\lambda^-)$ and solution procedure to problem (10) being similar to the properties of function $Q(\lambda^+)$ and solution procedure to problem (11), respectively, we only prove the properties of $Q(\lambda^-)$ and only show solution procedure to problem (10).

The parametric problem (10) has some special properties that can be utilized for solving the fractional geometric programming problem (8). Specifically, we show through following Lemma that the function $Q(\lambda^-)$ is convex, strictly monotonic decreasing and continuous, and then we show that nonlinear equation $Q(\lambda^-)=0$ has a unique solution $\lambda^-$ which is exactly the global optimal objective value of the problem (8).

**Lemma 3.2** Let $Q(\lambda^-)=max\{g_{1\alpha}^{-}(x)-\lambda^- g_2(x);x\in X\}$, then

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a. The function $Q(\lambda^-)$ is strictly monotonic decreasing and convex over $\mathsf{R}$.

b. The function $Q(\lambda^-)$ is continuous for $\lambda^- \in \mathsf{R}$.

c. $Q(\lambda^-) = 0$ has a unique solution.

**Proof.** (a) The monotonically decreasing of $Q(\lambda^-)$ follows from the positivity of $g_2(x)$. For convexity, let $0 \le t \le 1$ and $x_t$ be the optimal solution that maximizes $Q(t\lambda_1^- + (1-t)\lambda_2^-)$ with $\lambda_1^- \neq \lambda_2^-$. Then,

$$Q(t\lambda_1^- + (1-t)\lambda_2^-) = g_{1\alpha}^-(x_t) - (t\lambda_1^- + (1-t)\lambda_2^-)g_2(x_t)$$
$$= t[g_{1\alpha}^-(x_t) - \lambda_1^- g_2(x_t)] + (1-t)[g_{1\alpha}^-(x_t) - \lambda_2^- g_2(x_t)]$$
$$\le t\max\{g_{1\alpha}^-(x) - \lambda_1^- g_2(x); x \in X\} + (1-t)\max\{g_{1\alpha}^-(x) - \lambda_2^- g_2(x); x \in X\}$$
$$= tQ(\lambda_1^-) + (1-t)Q(\lambda_2^-).$$

(b) Let $x^1$ be the optimal solution of $Q(\lambda^{-1})$, then $g_2(x^1)$ is a positive constant and we also have

$$Q(\lambda^{-1}) = \max\{g_{1\alpha}^-(x) - \lambda^{-1} g_2(x); x \in X\} = g_{1\alpha}^-(x^1) - \lambda^{-1} g_2(x^1).$$

The function $Q(\lambda^-)$ is continuous, because it is monotonically decreasing (based on ($a$)), and also for every $\varepsilon > 0$, we can find a $\delta = \dfrac{\varepsilon}{g_2(x^1)} > 0$, such that for all $\lambda^- \in \mathsf{R}, 0 \le |\lambda^- - \lambda^{-1}| \le \delta$, we have

$$|Q(\lambda^{-1}) - Q(\lambda^-)| = |\{g_{1\alpha}^-(x^1) - \lambda^{-1} g_2(x^1)\} - \max\{g_{1\alpha}^-(x) - \lambda^- g_2(x); x \in X\}|$$
$$\le |\{g_{1\alpha}^-(x^1) - \lambda^{-1} g_2(x^1)\} - \{g_{1\alpha}^-(x^1) - \lambda^- g_2(x^1)\}|$$
$$= |(\lambda^- - \lambda^{-1})g_2(x^1)| = |(\lambda^- - \lambda^{-1})|\,|g_2(x^1)|,$$

since $g_2(x^1) > 0$ and $0 \le |\lambda^- - \lambda^{-1}| \le \delta = \dfrac{\varepsilon}{g_2(x^1)}$, we have $0 \le |Q(\lambda^{-1}) - Q(\lambda^-)| \le \varepsilon$.

(c) Since for $g_2(x) > 0$, $\lim_{\lambda^- \to -\infty} Q(\lambda^-) = +\infty$ and $\lim_{\lambda^- \to +\infty} Q(\lambda^-) = -\infty$. Furthermore, based on monotonically decreasing of $Q(\lambda^-)$, we can conclude that $Q(\lambda^-) = 0$ has a unique solution.

Now, we have the following Theorem for the equivalence between the parametric problem (10) and the fractional geometric programming problem (8).

**Theorem 3.3** The variable $x^*$ is a global optimal solution to the fractional geometric programming problem (8) if and only if $x^*$ is a global optimal solution to the parametric problem (10) with the parameter $\lambda^{-*}$ such that $Q(\lambda^{-*}) = 0$.

**Proof**. Let $x^* \in X$ be a global optimal solution of the parametric problem $(P_{\lambda^{-*}})$, then we have $g_{1\alpha}^-(x^*) - \lambda^{-*} g_2(x^*) = 0$ and $g_{1\alpha}^-(x) - \lambda^{-*} g_2(x) \le g_{1\alpha}^-(x^*) - \lambda^{-*} g_2(x^*) = 0, \forall x \in X$. Since $g_2(x) > 0$, we have $\lambda^{-*} = \dfrac{g_{1\alpha}^-(x^*)}{g_2(x^*)} \ge \dfrac{g_{1\alpha}^-(x)}{g_2(x)}, \forall x \in X$. Thus, $\lambda^{-*}$ is the maximum of the fractional geometric programming problem (8) and $x^*$ is the global optimal solution of it.

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Conversely, let $x^*$ be a global optimal solution of the fractional geometric programming problem (8) and $\lambda^{-*}$ be the optimal objective function value, so we have $\lambda^{-*} = \frac{g_{1\alpha}^-(x^*)}{g_2(x^*)} \geq \frac{g_{1\alpha}^-(x)}{g_2(x)}, \forall x \in X.$ Since $g_2(x) > 0$, we have $g_{1\alpha}^-(x) - \lambda^{-*} g_2(x) \leq g_{1\alpha}^-(x^*) - \lambda^{-*} g_2(x^*) = 0, \forall x \in X$. This implies that $x^*$ is the global optimal solution of the parametric problem (10).

As will be presented in the next subsection, Dinkelbach's iterative algorithm relies on the solution of a sequence of parametric subproblems $(P_{\lambda_n^-})$ in order to converge to the global optimal solution of the fractional geometric programming problem (8).

### 3.2.3 Iterative algorithm for solving parametric problem (10)

Dinkelbach's iterative algorithm [16] solves the parametric problem (10) by generating a sequences of $\lambda^-$ converging to $\lambda^{-*}$. The algorithm terminates once the objective value of the problem (10) becomes zero.

Based on the properties of the parametric problem (10), it is easy to see that $Q(\lambda^-) > 0 \Leftrightarrow \lambda^- < \lambda^{-*}$, $Q(\lambda^-) = 0 \Leftrightarrow \lambda^- = \lambda^{-*}$ and $Q(\lambda^-) < 0 \Leftrightarrow \lambda^- > \lambda^{-*}$.
Therefore, the solution of the problem (10) ends up with finding the root of equation $Q(\lambda^-) = max\{g_{1\alpha}^-(x) - \lambda^- g_2(x); x \in X\} = 0$. Although there are a number of root-finding algorithms for solving nonlinear equations, in this paper, we apply the Newton's method to solve the parametric problem (10).

In Newton's method [16, 36], $\lambda_{n+1}^-$ is defined by, $\lambda_{n+1}^- = \lambda_n^- - \frac{Q(\lambda_n^-)}{Q'(\lambda_n^-)}$. We can use the approximated subgradient [26] at point $\lambda_n^-$ to estimate the derivative, $Q'(\lambda_n^-) = \frac{dQ(\lambda_n^-)}{d\lambda_n^-} \approx -g_2(x_n^*)$, which is the negative value of the denominator evaluated at $x_n^*$, a global optimal solution of $max\{g_{1\alpha}^-(x) - \lambda_n^- g_2(x); x \in X\}$. Therefore, we have $\lambda_{n+1}^- = \lambda_n^- - \frac{Q(\lambda_n^-)}{-g_2(x_n^*)} = \lambda_n^- + \frac{g_{1\alpha}^-(x_n^*) - \lambda_n^- g_2(x_n^*)}{g_2(x_n^*)} = \frac{g_{1\alpha}^-(x_n^*)}{g_2(x_n^*)}$.

The full procedure of the Dinkelbach's algorithm based on the Newton's method for solving (10) is as follows:

*Step* 1: Choose arbitrary $x_0 \in X$ and set $\lambda_1^- = \frac{g_{1\alpha}^-(x_0)}{g_2(x_0)}$ or simply set $\lambda_1^- = 0$ and $n = 1$.

*Step* 2: Solve $Q(\lambda_n^-) = max\{g_{1\alpha}^-(x) - \lambda_n^- g_2(x); x \in X\} = 0$. Denote the optimal solution as $x_n^*$.

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*Step* 3: If $|Q(\lambda_n^-)| < \delta$ (optimality tolerance), stop and output $x_n^*$ as the optimal solution and $\lambda_n^-$ as optimal objective. If $|Q(\lambda_n^-)| \geq \delta$, let $\lambda_{n+1}^- = \frac{g_{1\alpha}^-(x_n^*)}{g_2(x_n^*)}$, update $n$ with $n+1$ and update $\lambda_n^-$ with $\lambda_{n+1}^-$. Go to Step 2.

### 3.3 Solution algorithm

We now summarize the proposed approach for solving the problem (3) with positive trapezoidal fuzzy coefficient in this work and construct a solution algorithm.

The basic steps of the algorithm are given below:

*Step* 0: Start with an initial level set $\alpha = \alpha^* = 0$.

*Step* 1: Convert problem FFPGP into it's nonfuzzy version $\alpha$ -FPGP.

*Step* 2: Rewrite problem (4) in the forms of two optimization problems (8) and (9).

*Step* 3: Change the problems (8) and (9) into the two equivalent parametric problems (10) and (11), respectively.

*Step* 4: Solve the problems (10) and (11) for finding the lower and upper bound of the $\alpha$ -optimal solution of problem (4) by using Dinkelbach's algorithm.

*Step* 5: Set $\alpha = (\alpha^* + step) \in [0,1]$ and go to Step 1.

*Step* 6: Repeat again the above procedure until the interval [0,1] is fully exhausted. Then, stop.

**Remark 3.4** It should be stated here that in the solution algorithm suggested above, a systematic variation of $\alpha$ -level set among the interval [0,1] will yield another optimal solution to problem (10) and the decision maker must determine this $\alpha$ -level set according to his desire.

## 4 Numerical example

In this section, a numerical example is given to illustrate the validity of the algorithm proposed in Section 3.

**Example 4.1** Consider the following FFPGP problem:

$$Max \quad \frac{(2,2.5,3.5,5)x_1^2 x_2^{-1} x_3 x_4^{-1} + (1,5,6,9)x_1 x_2^{0.3} x_3^{1.5}}{3x_2^{-2} x_3 x_4 + x_1^{-1} x_2^{-0.5} x_4}$$

$$s.t. \qquad (12)$$

$$1 \leq x_1, x_2 \leq 14, 0.1 \leq x_3 \leq 1, 1 \leq x_4 \leq 10.$$

By using $\alpha$ -cut of the fuzzy numbers coefficients, the FFPGP problem (12) can be converted to the following nonfuzzy $\alpha$ -FPGP problem:

$$Max \quad \frac{[0.5\alpha + 2, -1.5\alpha + 5]x_1^2 x_2^{-1} x_3 x_4^{-1} + [4\alpha + 1, -3\alpha + 9]x_1 x_2^{0.3} x_3^{1.5}}{3x_2^{-2} x_3 x_4 + x_1^{-1} x_2^{-0.5} x_4}$$

$$s.t. \qquad (13)$$

$$1 \leq x_1, x_2 \leq 14, 0.1 \leq x_3 \leq 1, 1 \leq x_4 \leq 10.$$

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According to problem (5) in Section 3, the $\alpha$-FPGP problem (13) can be transformed into the following form:

$$Max \quad \frac{(0.5\alpha+2)^{1-q}(-1.5\alpha+5)^{q}x_1^{2}x_2^{-1}x_3x_4^{-1}+(4\alpha+1)^{1-q}(-3\alpha+9)^{q}x_1x_2^{0.3}x_3^{1.5}}{3x_2^{-2}x_3x_4+x_1^{-1}x_2^{-0.5}x_4}$$

$$s.t. \tag{14}$$

$$1\le x_1,x_2\le 14,\ 0.1\le x_3\le 1,\ 1\le x_4\le 10,\ 0\le q\le 1.$$

Now by putting $q=0$ and $q=1$, the lower and upper bounds of the optimal solution of problem (14) will find by solving the following problems (15) and (16), respectively:

$$Max \quad \frac{(0.5\alpha+2)x_1^{2}x_2^{-1}x_3x_4^{-1}+(4\alpha+1)x_1x_2^{0.3}x_3^{1.5}}{3x_2^{-2}x_3x_4+x_1^{-1}x_2^{-0.5}x_4}$$

$$s.t. \tag{15}$$

$$1\le x_1,x_2\le 14,\ 0.1\le x_3\le 1,\ 1\le x_4\le 10.$$

$$Max \quad \frac{(-1.5\alpha+5)x_1^{2}x_2^{-1}x_3x_4^{-1}+(-3\alpha+9)x_1x_2^{0.3}x_3^{1.5}}{3x_2^{-2}x_3x_4+x_1^{-1}x_2^{-0.5}x_4}$$

$$s.t. \tag{16}$$

$$1\le x_1,x_2\le 14,\ 0.1\le x_3\le 1,\ 1\le x_4\le 10.$$

By using the parametric approach [16], the above problems (15) and (16) will take the following (17) and (18) forms, respectively:

$$Max \quad (0.5\alpha+2)x_1^{2}x_2^{-1}x_3x_4^{-1}+(4\alpha+1)x_1x_2^{0.3}x_3^{1.5}-\lambda^{-}(3x_2^{-2}x_3x_4+x_1^{-1}x_2^{-0.5}x_4)$$

$$s.t. \tag{17}$$

$$1\le x_1,x_2\le 14,\ 0.1\le x_3\le 1,\ 1\le x_4\le 10.$$

$$Max \quad (-1.5\alpha+5)x_1^{2}x_2^{-1}x_3x_4^{-1}+(-3\alpha+9)x_1x_2^{0.3}x_3^{1.5}-\lambda^{+}(3x_2^{-2}x_3x_4+x_1^{-1}x_2^{-0.5}x_4)$$

$$s.t. \tag{18}$$

$$1\le x_1,x_2\le 14,\ 0.1\le x_3\le 1,\ 1\le x_4\le 10.$$

The above problems (17) and (18) has been solved using Dinkelbach's algorithm and the results are reported in Table 1. The absolute optimality tolerance for the parametric algorithms are set as $\delta=10^{-3}$.

**Table 1.** Results for the Example 4.1

| $\alpha$-level set | α-optimal objective value $(\lambda^{-^*},\lambda^{+^*})$ | global α-optimal solution $(x_1^*,x_2^*,x_3^*,x_4^*)$ |
|---|---|---|
| $\alpha=0$ | (1712.4, 10120.44) | (14, 14, 1, 1) |
| $\alpha=0.24$ | (2623.7, 9327.1) | (14, 14, 1, 1) |
| $\alpha=0.53$ | (3724.85, 8368.45) | (14, 14, 1, 1) |
| $\alpha=0.86$ | (4977.81, 7277.62) | (14, 14, 1, 1) |
| $\alpha=1$ | (5509.4, 6814.8) | (14, 14, 1, 1) |

## 5 Conclusion

This paper has dealt with a fuzzified version of a fractional posynomial geometric programming problem in which fuzzy parameters are involved in the objective function. The algorithm presented here proposed a solution technique using a parametric approach for

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solving fuzzy fractional posynomial geometric programming FFPGP problem. Based on the obtained results in the last section, we conclude that using the proposed solution algorithm is useful to solve a FFPGP problem.

The advantages of the proposed procedure in this paper with respect to the other work on fractional programs is as follows. The proposed problem in this paper is a generalization of the fractional posynomial geometric programming problem where the coefficients of numerator of objective function are fuzzy numbers.

To our knowledge, this is the first algorithm that has been proposed for solving this problem. We believe this problem could be important for the future study of the fuzzy fractional optimization.

The denominator of objective function adopted in this paper is still the real-valued function. In the future research, we may extend to consider the both numerator and denominator of objective function as the fuzzy-valued functions.

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