International Journal of Applied Operational Research
Vol. 11, No. 4, pp. 1-23, Autumn 2023

Journal homepage: ijorlu.liau.ac.ir

Identify the status of returns to scale of the multi-period two-stage systems and efficiency decomposition

S. Esfidania, F. Hosseinzadeh Lotfi, Sh. Razavyan*

Received: 12 May 2023**;** **Accepted:** 14 September 2023

Abstract In the present world, calculating the efficiency (or the performance) of systems with an internal structure, such as two-stage systems, is principally imperative during multi-time periods. In the present approach, the traditional two-stage Data Envelopment Analysis (DEA) model is developed to a multi-period two-stage DEA model, which evaluates the overall and periods of efficiencies synchronously. This approach which is used, is not only alone incapable of having functional capacities under the assumption of variable returns to scale (VRS), but is also inattentive to the importance or magnitude of data in different periods. In this study, in order to surmount these shortcomings, we expand the existing approaches and introduce a generalized model to measure the overall efficiency of a multi-period two-stage system under the VRS assumption, wherein, the importance of data in time- periods is considered in a diverse manner. According to this generalized model, theorems are also being presented to determine the type of returns to scale (RTS) of both stages, as well as the system of the entire time periods and also each period. Finally, the real data of Taiwanese non-life insurance companies, which has been extracted from the extant literature, is used to illustrate the proposed approach.

Keyword: Data Envelopment Analysis (DEA), Two Stage System, Multi-Period, Multi-Stage System, Returns to Scale.

1 Introduction

Data Envelopment Analysis (DEA) is a mathematical programming for evaluating the relative efficiency of decision making units (DMUs), which has been introduced by Charnes, Cooper, and Rhodes [1]. Recently, in real life problems, there are some systems that have a two-stage structure. The traditional DEA measures the efficiency of this system regardless of its internal structure. Thereby, this is one of the disadvantages of the traditional DEA model. In order to compute the performance of such systems, some authors applied two-stage DEA methods. DMUs with a two- stage structure are used in many of the recent studies in the field of DEA. Seiford and Zhu [2] introduced a model for evaluating the two-stage system that evaluates the

* **Corresponding Author. (✉)**
E-mail: sh_razavyan@azad.ac.ir

S. Esfidania
Department of Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran

F. Hosseinzadeh Lotfi
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Sh. Razavyan
Department of Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran

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[DOI: 10.71885/ijorlu-2023-1-646]

overall efficiency of the system and performance of stages independently. In that, in this model, in general, the efficiency of the system may be proficient, whilst every stage is not. So, Kao and Hwang [3], proposed a model by considering the correlations of stages. This model measures the efficiencies of a two-stage system and the overall efficiency, which is the product of the efficiencies of stages. Thus, the system is efficient if and only if both the stages are efficient. In the inefficient stage, in order to achieve an efficient system, it is necessary to reduce the inputs or increase the outputs, or even modify the intermediate products. The Kao and Hwang model cannot calculate the efficiency of the systems under the VRS assumption. Hence, Chen et al. [4] presented the models to measure the constant returns to scale (CRS) and VRS efficiencies of two-stage systems. In their models, the overall efficiency of the system is expressed as a weighted average of the efficiencies of stages. Next, Wang and Chin [5] proposed a model to overcome the weakness of the Kao and Hwang model and calculate the VRS efficiency of two-stage systems. They also demonstrated that the efficiency obtained from the Chen et al. model is greater than that of the efficiency attained from the Kao and Hwang model.

Wu et al. [6] used an additive DEA approach to evaluate the efficiency of the new two-stage network structures with undesirable intermediate outputs. Fei et al. [7] also proposed some DEA models to measure the efficiencies of the two-stage process and its sub-processes, in the presence of undesirable outputs. Chu et al. [8] proposed a two-stage DEA model to obtain the eco-efficiency of the entire two-stage system in terms of the eco-efficiency analysis of the Chinese provincial-level regions. Despotis et al. [9] reviewed the additive and multiplicative efficiency decomposition methods and introduced a method to measure the unique and unbiased efficiency scores of two-stage systems. Based on the mentioned approaches, many models are introduced to measure the efficiency of two-stage systems with comprehensive structures. A model for measuring the efficiency of two-stage systems in presence of shared resources and undesirable outputs using the heuristic algorithm was also presented by Wu et al. [10]. Then, Guo and Zhu [11] converted the non-cooperative models of Wu et al. [10] into a linear program by utilizing the Charnes-Cooper transformation or conversion. Izadikhah et al. [12] also rendered a model for measuring the efficiency of two-stage systems with liberally distributed initial inputs and shared intermediate outputs. Also, Xueqin et al. [13] measured the efficiency of commercial banks based on two-stage DEA model in presence of undesirable output. Akbarian [14] proposed the new network DEA based on DEA-ratio. The novel classify of flexible and integer data in two-stage network DEA proposed by Hosseini Monfared et al. [15] (Refer to ([16-22])). All these models are focused on measuring the efficiency in a definite time period.

In varied applications, we need to compute the efficiency of systems in multiple time periods. Hence, many studies have been done to evaluate the efficiency of DMUs in different time periods. For example, Portela et al. [23] calculated the efficiency of DMUs in several time periods by employing the data average. Then, Kao and Liu [24] introduced the aggregate model, connected network model, and relational network model to calculate the multi-period efficiency of DMUs. The aggregate model is used in the aggregation of data; and is solved independently, to obtain the efficiency in a specific period. The connected network model calculates the efficiency of each period separately. Then, the efficiency of the period with the highest performance is recognized as the efficiency of the system. But in the relational network model, the constraints of each period are used and thus, after solving this model, the overall efficiency of the system and efficiency of each period are calculated. However, in this model, the total efficiency of the system is defined as the weighted average of periods. Razavi Hajiagha et al. [25] utilized the mean-variance criteria and presented a method for measuring

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the efficiency of multi-period systems. Bansal et al. [26] also measured the multi-period additive efficiency of systems with non-positive and undesirable data. Jahani Sayyad Noveiri et al. [27] introduced a DEA-based procedure to measure the multi-period efficiency of systems in the presence of undesirable outputs. Also, Tamatam et al. [28] used DEA technique and measured the efficiency of Indian banking industry over the period 2008–2017 (also refer to [29, 30]).

These models ignore the internal structure of systems. Therefore, Kao and Hwang [31] expanded the relational network model and proposed a multiplier model that is able to calculate the overall efficiency of a multi-period two-stage system and periods at the same time. But in this model, the importance of data at the different time periods is equally acclaimed. The fact arises that, in this model, the efficiencies under the VRS assumption cannot be computed. Esfidani et al. [32], proposed a new additive approach to measure the efficiency of multi-period two-stage systems under the CRS and VRS assumptions. In these approaches, the indexes are given to identify the progress or regress conditions of the system and stages from a period to another period. But these models do not have the capacity to determine the type of RTS of two-stage systems in each period and whole time periods. Subsequently, Tohidnia and Tohidi [33] proposed methods to measure the multi-period global cost efficiency in network systems when the price of exogenous inputs is identified in whole time periods. Zhou et.al [34] evaluated the efficiency of banking systems under uncertainty with multi-period three-stage structures. A novel multi-period DEA model for measuring the efficiencies of parallel and series systems was proposed by Esmaeilzadeh and Kazemi matin [35].

Numerous models have been presented that estimated RTS based on DEA models. Banker and Thrall [36] deployed the fractional DEA model and proposed an approach for estimating RTS. By using CCR and BCC models, Färe and Grosskopf [37] suggested an alternative approach to estimate RTS. An additive model approach for estimating RTS in imprecise DEA was also proposed by Khodabakhshi et al. [38] (Refer to ([39-47])). In order to evaluate the supply chains, Sharma and Yu [44] introduced multi-Echolen DEA VRS models; whilst for determining the RTS of an integer-value, a DEA model was presented by Taleb et.al [44]. Benicio and De Mello [48] also offered a different type of RTS in DEA, where, the possibility of the presence of a concave upward efficient frontier was taken under consideration.

The mentioned studies focus on evaluating the RTS of black box systems. Khaleghi et.al [49] determined the classification of the RTS and scale elasticity of two-stage systems based on contemplating the scale elasticity in each stage. Zhang and yang [50] introduced some models to determine the type of RTS in a two-stage system by utilizing the DEA envelopment models within a specific time. In fact, firstly, they solved the model to identify the efficient DMUs in the stages and the entire system. Then, two models for determining the RTS corresponding to each stage (and the whole system) are presented. Hence, their methods have a complex structure in specifying RTS. Also, Fakhr Mousavi et al. [51] presented a different non-radial procedure that measure the RTS and scale economies (SE) of network structures. Recently, many articles have been presented in various fields of data envelopment analysis. For example, Lexicographic decomposition strategy for the two-stage network DEA model is suggested by Yang and Fang [52]. A book in the field of uncertainty in data envelopment analysis is presented by Hosseinzadeh Lotfi et al. [53]. Also, Yu [54] suggested a non-desired output and non-radial DEA model for measuring industrial parks' green development performance. A non-desired output and non-radial DEA model for measuring industrial parks' green development performance is suggested by Yu [2023]. (Also, refer to [55-57])

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Unlike the standard DEA, network DEA imposes hurdles for measuring scale efficiency due to the fact that (i) overall efficiency aggregated by the stage or divisional technical efficiencies is highly non-linear and only solvable in a heuristic manner, or (ii) the overall efficiency which concerns exclusively inputs and outputs of a system is difficult to be decomposed into divisional efficiencies. In this paper, we establish a mathematical transformation to convert the corresponding non-linear programming problem into second order cone programming. The transformation is shown to be versatile in dealing with both CRS and VRS models under the two-stage network DEA.

So far, a multiplier model, for calculating the efficiency of a multi-period two-stage systems under a VRS assumption, has not been presented. Therefore, we seek to present an approach, which calculates the efficiency of the whole system as well as the stages of efficiency of multi-period two-stage systems in each period and whole time periods, without complexity; thereby, determining the RTS status of multi-period two-stage systems, simultaneously. For this purpose, being attentive to the significance of data in time periods, is a crucial issue. Hence, in this paper, we will propose a model which evaluates the efficiency of multi-period two-stage systems under the VRS assumption such that, the importance of data is considered diversely in different time periods; whereas, the overall efficiency of the system over the whole time periods is decomposed as a product of the efficiency of stages. In addition, the overall efficiency of system can be decomposed to the efficiency of stages in each period. On the whole, the efficiency of the system (or efficiency of stages) in whole time periods can be decomposed as a weighted average of the overall efficiency (or efficiency of stages in each period. Theorems are also being presented to determine the type of RTS of the full system and stages in each period, including whole time periods. Actually, on the basis of these theorems, if stages 1 and 2 comprise of IRS (or DRS or) CRS in each period (or whole time periods), the whole system will enclose IRS (or DRS), or CRS in each period (or whole time periods). We will also demonstrate that if one of the stages has IRS (or DRS) and another stages has CRS in each period (or whole time periods), the whole system shall include IRS (or DRS) in each period (or whole time periods). And if one of the stages has IRS and another stage has DRS in each period (or whole time periods), we shall present the theorem which identifies the condition of the RTS of the whole system in each period (or whole time periods). Finally, we can briefly state that by solving a model, we are able to specify the type of RTS in multi-period two-stage systems whilst calculating their efficiencies.

The rest of this paper is organized as follows: Initially, we introduce the structure of a two-stage system and review the two-stage DEA model proposed by Wang and Chin [5]. Next, by presenting the structure of a multi-period two-stage system, we shall review the model presented by Kao and Hwang [31]. In Section 2, we suggest a model to measure the multi-period two- stage system by considering the importance of data in a diverse manner in time periods in Section 3. The decompositions of the efficiencies are also proposed. In addition, we determine the type of the returns to scale of a multi-period two- stage system in each period and whole time periods. In Section 4, we present the results of a case study in Taiwanese non-life companies. Section 5 presents our conclusions and future research directions (guidelines).

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2 Preliminaries

Suppose there are n *DMUs* with two-stage structure that x_j, z_j, y_j are input, intermediate measure and output corresponding to each $DMU_j\ (j=1,\ldots,n)$. The structure of a two-stage system is depicted in Figure 1.

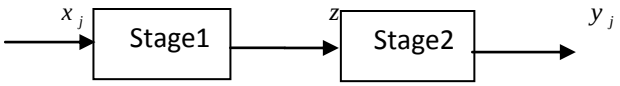


Fig. 1 Two-stage production system

Wang and Chin [5] proposed the following model which calculates the VRS efficiency of two-stage systems at a certain time period:

$$E_o = \max \quad [E_o^{\mathrm{I}} \times E_o^{\mathrm{II}} = \frac{\sum_{r=1}^{s} \mu_r y_{ro} + u^2}{\sum_{i=1}^{m} \omega_i x_{io} - u^1}]$$

$$s.t \quad \frac{\sum_{r=1}^{s} \pi_d z_{do}}{\sum_{i=1}^{m} \omega_i x_{io} - u^1} \le 1, \qquad j=1,...,n \qquad (1)$$

$$\frac{\sum_{r=1}^{s} \mu_r y_{ro} + u^2}{\sum_{r=1}^{s} \pi_d z_{do}} \le 1, \qquad j=1,...,n$$

$$\omega_i, \mu_r, \pi_d \ge 0, \qquad i=1,...,m \quad r=1,...,s \quad d=1,...,D$$

$$u^1, u^2 \ \ free$$

Note that ω_i, μ_r, π_d are the weights of inputs, outputs and intermediate measures corresponding to a two-stage system. Now, suppose we have *n* DMUs with two-stage structure and there are *q* time periods. Figure 2 shows the structure of a multi-period two-stage system. In time period *p*, each $DMU_j\ (j=1,\ldots,n)$ in Stage 1, consumes inputs $x_{ij}^p; i=1,...,m$ to produce intermediate products $z_{dj}^p; d=1,...,\mathrm{D}$, and then in Stage 2, consumes the intermediate products $z_{dj}^p; d=1,...,\mathrm{D}$ to produce outputs $y_{rj}^p; r=1,...,\mathrm{s}$. We consider $x_{ij} = \sum_{p=1}^{q} x_{ij}^{\ p}$, $z_{dj} = \sum_{p=1}^{q} z_{dj}^{\ p}$, $\mathrm{y}_{rj} = \sum_{p=1}^{q} y_{rj}^{\ p}$ as the aggregate inputs, intermediate products and outputs of DMU_j, respectively.

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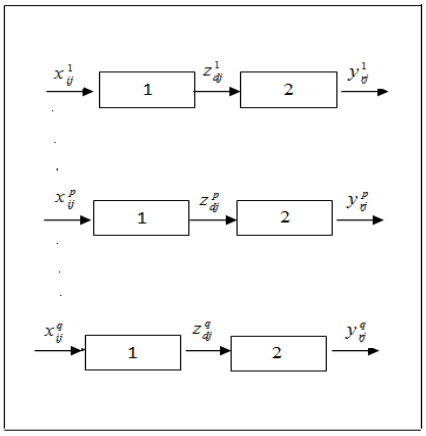


Fig. 2 Multi-period two-stage system

In order to evaluate the efficiency of this system, Kao and Hwang [31] suggested the following model that measures the efficiency of a system and stages in each time period and whole time periods, simultaneously:

$$
\begin{aligned}
& E_o^s = \max \quad \sum_{r=1}^{s} \mu_r y_{ro} \\
& s.t \\
& \sum_{i=1}^{m} \omega_i x_{io} = 1 \\
& \sum_{d=1}^{D} \pi_d z_{dj} - \sum_{i=1}^{m} \omega_i x_{ij} \le 0, \qquad j = 1,\ldots..,n \\
& \sum_{d=1}^{D} \pi_d z_{dj}^p - \sum_{i=1}^{m} \omega_i x_{ij}^p \le 0, \qquad j = 1,\ldots.,n \qquad p = 1,\ldots,q \qquad (2) \\
& \sum_{r=1}^{s} \mu_r y_{rj} - \sum_{d=1}^{D} \pi_d z_{dj} \le 0, \qquad j = 1,\ldots..,n \\
& \sum_{r=1}^{s} \mu_r y_{rj}^p - \sum_{d=1}^{D} \pi_d z_{dj}^p \le 0, \qquad j = 1,\ldots.,n \qquad p = 1,\ldots,q \\
& \mu_r, \pi_d, \omega_i \ge 0 \quad r = 1,\ldots,s \qquad r = 1,\ldots,S \qquad d = 1,\ldots,D \qquad i = 1,\ldots,m
\end{aligned}
$$

3 The proposed generalized model

In this section firstly, we will suggest a model to measure the VRS efficiency of multi-period two-stage systems. At the end, some definitions and theorems are given to identify the type of the RTS of system and stages in each period p and whole time periods. Note that, in model (2), the data has equal importance in different time periods. Actually, the weights of inputs (ω_i), the weights of outputs (μ_r) and the weights of intermediate products weights (π_d) are similar in all periods; and in many situations it is also necessary to calculate the VRS efficiency and identify the status of RTS corresponding to the whole system and stages in each of the time periods as well as the whole time periods. In this case, this model is not applicable. Thus, we modify this model and use model (1) to introduce a model to overcome these shortcomings. Actually, we perform VRS efficiency calculation and identify the RTS

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status of multi-period two-stage systems by solving a model. Hence, by considering the input weight ω_i^p, the output weight μ_r^p and the intermediate products weight π_d^p to time period time *p*, the generalized model is given as follows:

$$E_o^s = \max \quad \frac{\sum_{p=1}^{q}\sum_{r=1}^{s}\mu_r^p y_{ro}^p + \sum_{p=1}^{q}u^{2p}}{\sum_{p=1}^{q}\sum_{i=1}^{m}\omega_i^p x_{io}^p - \sum_{p=1}^{q}u^{1p}}$$

$$s.t \quad \frac{\sum_{p=1}^{q}\sum_{d=1}^{D}\pi_d^p z_{dj}^p}{\sum_{p=1}^{q}\sum_{i=1}^{m}\omega_i^p x_{ij}^p - \sum_{p=1}^{q}u^{1p}} \le 1, \quad j=1,\ldots,n \qquad (3)$$

$$\frac{\sum_{d=1}^{D}\pi_d^p z_{dj}^p}{\sum_{i=1}^{m}\omega_i^p x_{ij}^p - u^{1p}} \le 1, \qquad j=1,\ldots,n \qquad p=1,\ldots,q$$

$$\frac{\sum_{p=1}^{q}\sum_{r=1}^{s}\mu_r^p y_{rj}^p + \sum_{p=1}^{q}u^{2p}}{\sum_{p=1}^{q}\sum_{d=1}^{D}\pi_d^p z_{dj}^p} \le 1, \quad j=1,\ldots,n$$

$$\frac{\sum_{r=1}^{s}\mu_r^p y_{rj}^p + u^{2p}}{\sum_{d=1}^{D}\pi_d^p z_{dj}^p} \le 1, \qquad j=1,\ldots,n \qquad p=1,\ldots,q$$

$$\omega_i^p, \mu_r^p, \pi_d^p \ge 0, \qquad i=1,\ldots,m \qquad r=1,\ldots,s \qquad d=1,\ldots,D \qquad p=1,\ldots,q$$

$$u^{1p}, u^{2p} \quad free \qquad p=1,\ldots,q$$

This model can be converted to the linear form, easily. By solving model (3), the efficiencies of the system in each period p and over q time periods are obtained as follows:

$$E_o^s = \frac{\sum_{p=1}^{q}\sum_{r=1}^{s}\mu_r^{p^*} y_{ro}^p + \sum_{p=1}^{q}u^{2p^*}}{\sum_{p=1}^{q}\sum_{i=1}^{m}\omega_i^{p^*} x_{io}^p - \sum_{p=1}^{q}u^{1p^*}}, \qquad E_o^{\mathrm{I}} = \frac{\sum_{p=1}^{q}\sum_{d=1}^{D}\pi_d^{p^*} z_{do}^p}{\sum_{p=1}^{q}\sum_{i=1}^{m}\omega_i^{p^*} x_{io}^p - \sum_{p=1}^{q}u^{1p^*}}, \qquad E_o^{\mathrm{II}} = \frac{\sum_{p=1}^{q}\sum_{r=1}^{s}\mu_r^{p^*} y_{ro}^p + \sum_{p=1}^{q}u^{2p^*}}{\sum_{p=1}^{q}\sum_{d=1}^{D}\pi_d^{p^*} z_{do}^p}$$

$$E_o^{s(p)} = \frac{\sum_{r=1}^{s}\mu_r^{p^*} y_{ro}^p + \mathrm{u}^{2p^*}}{\sum_{i=1}^{m}\omega_i^{p^*} x_{io}^p - u^{1p^*}}, \qquad E_o^{\mathrm{I}(p)} = \frac{\sum_{d=1}^{D}\pi_d^{p^*} z_{do}^p}{\sum_{i=1}^{m}\omega_i^{p^*} x_{io}^p - u^{1p^*}}, \qquad E_o^{\mathrm{II}(p)} = \frac{\sum_{r=1}^{s}\mu_r^{p^*} y_{ro}^p + \mathrm{u}^{2p^*}}{\sum_{d=1}^{D}\pi_d^{\mathrm{p}^*} z_{do}^p}$$

Wherein, E_o^s, E_o^{I} and E_o^{II} demonstrate the overall efficiency of the system and efficiency of stage 1 and the efficiency of Stage 2 during the whole time period *q* respectively. In a similar way, $E_o^{s(p)}, E_o^{\mathrm{I}(p)}$ and $E_o^{\mathrm{II(p)}}$ give the overall efficiency and efficiencies of stages 1 and 2 in time period *p* , respectively.

Theorem 1. It is noted that DMU_o is overall efficient if and only if $E_o^s = 1$. And also, DMU_o is efficient in stage 1, 2 over q time periods if and only if $E_o^{\mathrm{I}} = E_o^{\mathrm{II}} = 1$.

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3.1 The decompositions of efficiencies

We can decompose the overall efficiency of the system to the product of the efficiencies of stages, in each of the periods, p and over q time periods. Now, in order to show the relationships between the overall efficiency of the system in each period p and whole time periods, including the correlations between the efficiency of stages over the q time period and efficiency of stages in each period p, we introduce the following relationships:

$E_o^s = E_o^{\mathrm{I}} \times E_o^{\mathrm{II}}$, $E_o^{s(p)} = E_o^{\mathrm{I}(p)} \times E_o^{\mathrm{II(p)}}$

Therefore, we can decompose the overall efficiency of system to product of the efficiencies of stages in each period p and over q time periods. Now, in order to show the relationships between the overall efficiency of the system in each period p and whole time periods, and the relationships between the efficiency of stages over q time period and efficiency of stages in each period p, we introduce the following relationships:

$$E_o^s = \sum_{p=1}^{q} \varpi^p E_o^{s(p)}, \qquad \varpi^p = \frac{\sum_{i=1}^{m} \omega_i^{\mathrm{p}*} x_{io}{}^p - u^{1p*}}{\sum_{p=1}^{q}\sum_{i=1}^{m} \omega_i^{\mathrm{p}*} x_{io}{}^p - \sum_{p=1}^{q} u^{1p*}}$$

$$E_o^{\mathrm{I}} = \sum_{p=1}^{q} \varpi^{\mathrm{I}(p)} E_o^{\mathrm{I}(p)}, \qquad \varpi^{\mathrm{I}(p)} = \frac{\sum_{i=1}^{m} \omega_i^{\mathrm{p}*} x_{io}{}^p - u^{1p*}}{\sum_{p=1}^{q}\sum_{i=1}^{m} \omega_i^{\mathrm{p}*} x_{io}{}^p - \sum_{p=1}^{q} u^{1p*}}$$

$$E_o^{\mathrm{II}} = \sum_{p=1}^{q} \varpi^{\mathrm{II}(p)} E_o^{\mathrm{II}(p)}, \qquad \varpi^{\mathrm{II}(p)} = \frac{\sum_{d=1}^{D} \pi_d^{\mathrm{p}*} z_{do}^{\,p}}{\sum_{p=1}^{q}\sum_{d=1}^{D} \pi_d^{\mathrm{p}*} z_{do}^{\,p}}$$

$\varpi^{\mathrm{II}(p)}$ can be considered by the proportion of the aggregated input consumed by stage 2 in period to the total aggregated inputs consumed by stage2 over whole time periods. It can be concluded that the weighted average of the efficiencies of periods is defined as the overall efficiency of the system over q time period. Note that, decomposition of the efficiencies can be done as follows:

$$E_o^s = E_o^{\mathrm{I}} \times E_o^{\mathrm{II}} = (\sum_{p=1}^{q} \varpi^{\mathrm{I}(p)} E_o^{\mathrm{I}(p)})(\sum_{p=1}^{q} \varpi^{\mathrm{II}(p)} E_o^{\mathrm{II}(p)})$$

$$E_o^s = \sum_{p=1}^{q} \varpi^p E_o^{s(p)} = \sum_{p=1}^{q} \varpi^p (E_o^{\mathrm{I}(p)} * E_o^{\mathrm{II}(p)})$$

Finally, by using the above decompositions, we get the following theorem:

Theorem 2. The multi-period two-stage system is efficient under the VRS assumption, if and only if its stages are efficient under the VRS assumption in the all periods.

In the cases where, the optimal solution of the model (3) may not be unique or distinctive, the overall efficiency decomposition will not be characteristic. We shall also not be able to compare the efficiency of different DMUs together. So, we follow the Kao, Hwang([3], [31]) approaches here, to find a set of multipliers (coefficients) for this period and compare the efficiency of DMUs in a period, which produces the largest efficiency, while in general, the efficiency of the system is unchanged. This approach is persistent for other periods. Note that each period has a two-stage structure and we assume that the efficiency of period t, is the most important period. Thus, if the first is stage 1, in terms of period t, it is more important than the second stage of the same period, in order to calculate the maximum efficiency of

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Stage 1, the following model is solved, while the efficiency of period t and the overall efficiency remain unaltered:

$$
\begin{aligned}
E_o^{\mathrm{I}(t)} = \max\ & \frac{\sum_{d=1}^{D} \pi_d^t z_{do}^t}{\sum_{i=1}^{m} \omega_i^t x_{io}^t - u^{1t}} \\
s.t\ & \frac{\sum_{d=1}^{D} \pi_d^t z_{dj}^t}{\sum_{i=1}^{m} \omega_i^t x_{ij}^t - u^{1t}} \le 1, \qquad j = 1,...,n \qquad (4) \\
& \frac{\sum_{r=1}^{s} \mu_r^t y_{rj}^t + u^{2t}}{\sum_{d=1}^{D} \pi_d^t z_{dj}^t} \le 1, \qquad j = 1,...,n \\
& \frac{\sum_{p=1}^{q}\sum_{r=1}^{s} \mu_r^p y_{ro}^p + \sum_{p=1}^{q} u^{2p}}{\sum_{p=1}^{q}\sum_{i=1}^{m} \omega_i^p x_{io}^p - \sum_{p=1}^{q} u^{1p}} = E_o^s \\
& \frac{\sum_{r=1}^{s} \mu_r^t y_{ro}^t + u^{2t}}{\sum_{i=1}^{m} \omega_i^t x_{io}^t - u^{1t}} = E_o^{s(t)} \\
& \omega_i^p, \mu_r^p, \pi_d^p \ge 0, \qquad i = 1,...,m \quad r = 1,...,s \quad d = 1,...,D \quad p = 1,...,q \\
& u^{1p}, u^{2p} \ \ free \qquad p = 1,...,q
\end{aligned}
$$

Then, the efficiency of stage 2 is $E_o^{\mathrm{II}(t)} = E_o^{s(t)} / E_o^{\mathrm{I}(t)}$. Finally, we note that in model (3), if $u^{1p} = u^{1p} = 0,\ p = 1,...,q$, this model is converted to the Kao and Hwang model [31].

3.2 Determining RTS in multi-period two-stage systems

In this subsection, firstly, we will introduce the basic definitions of production possibility set (PPS) and RTS corresponding to the multi-period two-stage systems. Then, theorems are presented to identify the type of RTS of whole system and stages in each period p and over q time periods. In following Banker et al. [51], we primarily define the PPS of Stage 1 in period $p(T_{1p})$, of Stage 2 in period $p(T_{2p})$, of the system in period $p(T_p)$, of Stage 1 in the whole time periods (T_1), of Stage 2 in the whole time periods (T_2) and of the system of Stage 1, in the whole time periods (T), as follows:

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[DOI: 10.71885/ijorlu-2023-1-646]

$$T_{1p}=\left\{(\mathrm{x}^p,\mathrm{z}^p)\middle|\mathrm{X}^p\lambda^p\le\mathrm{x}^p,\ \mathrm{Z}^p\lambda^p\ge\mathrm{z}^p,\ 1\lambda^p=1,\ \lambda^p\ge0\right\}$$

$$T_{2p}=\left\{(\mathrm{z}^p,\mathrm{y}^p)\middle|\mathrm{Z}^p\gamma^p\le\mathrm{z}^p,\ \mathrm{Y}^p\gamma^p\ge y^p,\ 1\gamma^p=1,\ \gamma^p\ge0\right\}$$

$$T_p=\left\{(\mathrm{x}^p,\mathrm{z}^p,\mathrm{y}^p)\middle|\mathrm{X}^p\lambda^p\le\mathrm{x}^p,\mathrm{Z}^p\lambda^p\ge\mathrm{z}^p,\mathrm{Z}^p\gamma^p\le\mathrm{z}^p,\mathrm{Y}^p\gamma^p\ge y^p,\ 1\lambda^p=1,\ 1\gamma^p=1,\lambda^p\ge0,\ \gamma^p\ge0\right\}$$

$$T_1=\left\{\underset{p=1,...,q}{(\mathrm{x}^p,\mathrm{z}^p)}\middle|\mathrm{X}^p\lambda^p\le\mathrm{x}^p,\ \mathrm{Z}^p\lambda^p\ge\mathrm{z}^p,\ 1\lambda^p=1,\ \lambda^p\ge0,\ p=1,...,q\right\}$$

$$T_2=\left\{\underset{p=1,...,q}{(\mathrm{z}^p,\mathrm{y}^p)}\middle|\mathrm{Z}^p\gamma^p\le\mathrm{z}^p,\ \mathrm{Y}^p\gamma^p\ge y^p,\ 1\gamma^p=1,\ \gamma^p\ge0,\ p=1,...,q\right\}$$

$$T=\left\{\underset{p=1,...,q}{(\mathrm{x}^p,\mathrm{z}^p,\mathrm{y}^p)}\middle|\mathrm{X}^p\lambda^p\le\mathrm{x}^p,\ \mathrm{Z}^p\lambda^p\ge\mathrm{z}^p,\mathrm{Z}^p\gamma^p\le\mathrm{z}^p,\ \mathrm{Y}^p\gamma^p\ge y^p,\ 1\lambda^p=1,\ 1\gamma^p=1,\lambda^p\ge0,\gamma^p\ge0,p=1,...,q\right\}$$

Where, $T_1\cup T_2=\bigcup_{p=1}^{q}T_p=T$, $\bigcup_{p=1}^{q}T_{1p}=T_1$, $\bigcup_{p=1}^{q}T_{2p}=T_2$.

The following definitions are introduced for detecting the type of RTS in whole system (or Stage 1, or Stage 2) in a certain time period p.

Definition1. Suppose DMU_o is efficient in the whole system and $T=T_p$. Then,

1-1. DMU_o in the whole system has increasing returns to scale (*IRS*), if there is $\delta^*>0$, such that:

$\forall\delta\in(0,\delta^*]\Rightarrow Z_\delta=((1+\delta)x_o,(1+\delta)z_o,(1+\delta)y_o)\in\mathrm{int}T$

1-2. DMU_o in the whole system (has decreasing returns to scale (*DRS*), if there is $\delta^{'*}>0$, such that:

$\forall\delta^{'}\in(0,\delta^{'*}]\Rightarrow Z_{\delta^{'}}=((1-\delta^{'})x_o,(1-\delta^{'})z_o,(1-\delta^{'})y_o)\in\mathrm{int}T$.

1-3. DMU_o in the whole system has constant returns to scale (*CRS*), if there are $\delta^*>0$, $\delta^{'*}>0$, such that at least one of the following conditions holds:

$$1-3-1.\begin{cases}\forall\delta\in(0,\delta^*]\Rightarrow Z_\delta=((1+\delta)x_o,(1+\delta)z_o,(1+\delta)y_o)\ \textit{is efficient}\\ \forall\delta^{'}\in(0,\delta^{'*}]\Rightarrow Z_{\delta^{'}}=((1-\delta^{'})x_o,(1-\delta^{'})z_o,(1-\delta^{'})y_o)\textit{is efficient}\end{cases}$$

$$1-3-2.\begin{cases}\forall\delta\in(0,\delta^*]\Rightarrow Z_\delta=((1+\delta)x_o,(1+\delta)z_o,(1+\delta)y_o)\notin T\\ \forall\delta^{'}\in(0,\delta^{'*}]\Rightarrow Z_{\delta^{'}}=((1-\delta^{'})x_o,(1-\delta^{'})z_o,(1-\delta^{'})y_o)\notin\mathrm{T}\end{cases}$$

$$1-3-3.\begin{cases}\forall\delta\in(0,\delta^*]\Rightarrow Z_\delta=((1+\delta)x_o,(1+\delta)z_o,(1+\delta)y_o)\ \textit{is efficient}\\ \forall\delta^{'}\in(0,\delta^{'*}]\Rightarrow Z_{\delta^{'}}=((1-\delta^{'})x_o,(1-\delta^{'})z_o,(1-\delta^{'})y_o)\notin\mathrm{T}\end{cases}$$

$$1-3-3.\begin{cases}\forall\delta\in(0,\delta^*]\Rightarrow Z_\delta=((1+\delta)x_o,(1+\delta)z_o,(1+\delta)y_o)\notin\mathrm{T}\\ \forall\delta^{'}\in(0,\delta^{'*}]\Rightarrow Z_{\delta^{'}}=((1-\delta^{'})x_o,(1-\delta^{'})z_o,(1-\delta^{'})y_o)\textit{is efficient}\end{cases}$$

Also, similar Definitions can be given for the stage 1 with $T=T_{1p}$, $Z_\delta=((1+\delta)x_o,(1+\delta)z_o)$, $Z_{\delta^{'}}=((1-\delta^{'})x_o,(1-\delta^{'})z_o)$ and stage 2 with $T=T_{2p}$, $Z_\delta=((1+\delta)z_o,(1+\delta)y_o)$, $Z_{\delta^{'}}=((1-\delta^{'})z_o,(1-\delta^{'})y_o)$. Therefore, based on the optimal solutions of model (3), we presented theorems to identify the type of RTS in a multi-period two-stage system. By using the status of the sign u^{1p^*} in period p and Definition 1, theorem 3 is rendered to recognize the type of RTS of DMU_o in stage 1.

Theorem3. Suppose DMU_o is efficient in stage 1 of period p, i.e $E_o^{1(p)}=1$, then

3-i. If $u^{1p^*}>0$, DMU_o in stage 1 of period p has *IRS*.

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Proof . Suppose $(\mu^{(p)*},\omega^{(p)*},\pi^{(p)*},u^{1p*},u^{2p*})$ is the optimal solution of model (3) and there is $\delta^{p*}>0$ such that $Z^{p}{}_{\delta^{p}}=((1+\delta^{p})x_o^{p},(1+\delta^{p})z_o^{p}),\ \delta^{p}\in(0,\delta^{p*}]$. Since DMU_o is efficient in stage 1 of period p, we have: $\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p}+u^{1p*}=0$. In this case,

$$P^{p}{}_{1\delta^{p}}=\pi^{(p)*}(1+\delta^{p})z_o^{p}-\omega^{(p)*}(1+\delta^{p})x_o^{p}+u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p})(1+\delta^{p})+$$
$$u^{1p*}+\delta^{p}u^{1p*}-\delta^{p}u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p}+u^{1p*})(1+\delta^{p})-\delta^{p}u^{1p*}=0-\delta^{p}u^{1p*}=-\delta^{p}u^{1p*}$$

If $u^{1p*}>0$, we have $P^{p}{}_{1\delta^{p}}<0$. Therefore, $Z^{p}{}_{\delta^{p}}\in \mathrm{int}T_{1p}$ and according to Definition 1, DMU_o in stage 1 of period p has *IRS* and the proof is complete.

3-ii. If $u^{1p*}<0$, DMU_o in stage 1 of period p has *DRS*.

Proof. Suppose $(\mu^{(p)*},\omega^{(p)*},\pi^{(p)*},u^{1p*},u^{2p*})$ is the optimal solution of model (3), and for period p, there is $\delta^{'p*}>0$ such that, $Z^{p}{}_{\delta^{'p}}=((1-\delta^{'p})x_o^{p},(1-\delta^{'p})z_o^{p}),\ \delta^{'p}\in(0,\delta^{'p*}]$. Since DMU_o is efficient in stage 1 of period p, we have: $\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p}+u^{1p*}=0$. In this case,

$$P^{p}{}_{1\delta^{'p}}=\pi^{(p)*}(1-\delta^{'p})z_o^{p}-\omega^{(p)*}(1-\delta^{'p})x_o^{p}+u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p})(1-\delta^{'p})+$$
$$u^{1p*}+\delta^{'p}u^{1p*}-\delta^{'p}u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p}+u^{1p*})(1-\delta^{'p})+\delta^{'p}u^{1p*}=0+\delta^{'p}u^{1p*}=\delta^{'p}u^{1p*}$$

Thus, if $u^{1p*}<0$, we have $P^{p}{}_{1\delta^{'p}}<0$. Therefore, $Z^{p}{}_{\delta^{'p}}\in \mathrm{int}T_{1p}$, and according to Definition 1, DMU_o in stage 1 of period p has *DRS*.

3-iii. If $u^{1p*}=0$, DMU_o in stage 1 of period p has *CRS*.

Proof. Suppose $(\mu^{(p)*},\omega^{(p)*},\pi^{(p)*},u^{1p*},u^{2p*})$ is the optimal solution of model (3) and there are $\delta^{p*}>0$, $\delta^{'p*}>0$, such that

$$\begin{cases} Z^{p}{}_{\delta^{p}}=((1+\delta^{p})x_o^{p},(1+\delta^{p})z_o^{p}),\ \delta^{p}\in(0,\delta^{p*}] \\ Z^{p}{}_{\delta^{'p}}=((1-\delta^{'p})x_o^{p},(1-\delta^{'p})z_o^{p}),\delta^{'p}\in(0,\delta^{'p*}] \end{cases}$$

Since DMU_o is efficient in stage 1 of period p, we have: $\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p}+u^{1p*}=0$. In this case,

$$\begin{cases} P^{p}{}_{1\delta^{p}}=\pi^{(p)*}(1+\delta^{p})z_o^{p}-\omega^{(p)*}(1+\delta^{p})x_o^{p}+u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p})(1+\delta^{p})+ \\ u^{1p*}+\delta^{p}u^{1p*}-\delta^{p}u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p}+u^{1p*})(1+\delta^{p})-\delta^{p}u^{1p*}=0 \\ \\ P^{p}{}_{1\delta^{'p}}=\pi^{(p)*}(1-\delta^{'p})z_o^{p}-\omega^{(p)*}(1-\delta^{'p})x_o^{p}+u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p})(1-\delta^{'p})+ \\ u^{1p*}+\delta^{'p}u^{1p*}-\delta^{'p}u^{1p*}=(\pi^{(p)*}z_o^{p}-\omega^{(p)*}x_o^{p}+u^{1p*})(1-\delta^{'p})+\delta^{'p}u^{1p*}=0 \\ P^{p}{}_{1\delta^{p}}=P^{p}{}_{1\delta^{'p}}=0 \end{cases}$$

Thus, if $u^{1p*}=0$, then, $Z^{p}{}_{\delta^{p}},Z^{p}{}_{\delta^{'p}}$ are efficient in stage 1 of period p and according to Definition 1, DMU_o in stage 1 of period p has *CRS*.

In whole time periods, based on Definition 1 and the sign status of $\sum_{p=1}^{q}u^{1p*}$, we can identify the type of RTS DMU_o in stage 1. For this reason, we present the following theorem:

Theorem4. Suppose DMU_o is efficient in stage 1 over q time periods, i.e $E_o^{\mathrm{I}}=1$,

4-i. If $\sum_{p=1}^{q}u^{1p*}>0$, DMU_o in stage 1 of q time period has *IRS*.

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[DOI: 10.71885/ijorlu-2023-1-646]

Proof. Suppose $(\mu^{(p)*}, \omega^{(p)*}, \pi^{(p)*}, u^{1p*}, u^{2p*})$ is the optimal solution of model (3), and corresponding to each period p, $(p=1,...,q)$, there are $\delta^{p*}>0,\ p=1,...,q$ such that:
$Z^{p}_{\delta^{p}} = ((1+\delta^{p})x_o^p, (1+\delta^{p})z_o^p)$, $\delta^{p} \in (0, \delta^{p*}];\ p=1,...,q$. Since DMU_o is efficient in Stage 1 over q time periods, we have:
$w^{(p)*} z_o^p - v^{(p)*} x_o^p + u^{1p*} = 0$, $p=1,...,q$
In this case,

$$\begin{cases} P^{p}_{1_{\delta^p}} = \pi^{(p)*}(1+\delta^p) z_o^p - \omega^{(p)*}(1+\delta^p) x_o^p + u^{1p*} = (\pi^{(p)*} z_o^p - \omega^{(p)*} x_o^p)(1+\delta^p) + \\ u^{1p*} + \delta^p u^{1p*} - \delta^p u^{1p*} = (\pi^{(p)*} z_o^p - \omega^{(p)*} \mathrm{x}_o + u^{1p*})(1+\delta^p) x_o^p - \delta^p u^{1p*} = \\ 0 - \delta^p u^{1p*} = -\delta^p u^{1p*},\ p=1,...,q \quad \Rightarrow \sum_{p=1}^{q} P^{p}_{1_{\delta^p}} = \sum_{p=1}^{q} -\delta^p u^{1p*} = -\sum_{p=1}^{q} \delta^p u^{1p*} \end{cases}$$

Therefore, if $\sum_{p=1}^{q} u^{1p*} > 0$, then $\sum_{p=1}^{q} P^{p}_{1_{\delta^p}} < 0$. Thus, where $\delta^{p} \in (0, \delta^{p*}]$, $p=1,...,q$. This means that according to Definition 1, DMU_o has IRS.

4-ii. If $\sum_{p=1}^{q} u^{1p*} < 0$, DMU_o in stage 1 of q time period has DRS.

Proof. It is similar to the proof of **(4-i).**

4-iii. If $\sum_{p=1}^{q} u^{1p*} = 0$, DMU_o in stage 1 of q time period has CRS.

Proof. It is similar to the proof of **(4-i).**

Theorem5 determines the type of RTS DMU_o in stage 2, by using the sign u^{2p*} in period p:

Theorem5. Suppose DMU_o is efficient in stage 2 of period p, i.e $E_o^{\mathrm{II}(p)} = 1$, then

5-i. If $u^{2p*} > 0$, DMU_o in stage 2 of period p has IRS.

Proof. Suppose $(\mu^{(p)*}, \omega^{(p)*}, \pi^{(p)*}, u^{1p*}, u^{2p*})$ is the optimal solution of model (3) and there is $\delta^{p*}>0$ and $Z^{p}_{\delta^{p}} = ((1+\delta^{p})z_o^p, (1+\delta^{p})y_o^p)$, $\delta^{p} \in (0, \delta^{p*}]$. Since DMU_o is efficient in Stage 2 of period p, we have $\mu^{(p)*} y_o^p - \pi^{(p)*} z_o^p + u^{2p*} = 0$. In this case,

$P^{p}_{2_{\delta^p}} = \mu^{(p)*}(1+\delta^p) y_o^p - \pi^{(p)*}(1+\delta^p) z_o^p + u^{1p*} = (\mu^{(p)*} y_o^p - \pi^{(p)*} z_o^p)(1+\delta^p) + u^{2p*} +$
$\delta^p u^{2p*} - \delta^p u^{2p*} = (\mu^{(p)*} y_o^p - \pi^{(p)*} z_o^p + u^{2p*})(1+\delta^p) - \delta^p u^{2p*} = 0 - \delta^p u^{2p*} = -\delta^p u^{2p*}$

Thus, if $u^{2p*} > 0$, then $P^{p}_{2_{\delta^p}} < 0$. $Z^{p}_{\delta^p} \in \mathrm{int}\, T_{2p}$ and according to Definition 1, DMU_o in stage 2 of period p has IRS.

5-ii. If $u^{2p*} < 0$, DMU_o in stage 1 of period p has DRS.

Proof. Suppose $(\mu^{(p)*}, \omega^{(p)*}, \pi^{(p)*}, u^{1p*}, u^{2p*})$ is the optimal solution of model (3), and for period p, there is $\delta^{'p*}>0$ and $Z^{p}_{\delta^{'p}} = ((1-\delta^{'p})z_o^p, (1-\delta^{'p})y_o^p)$, $\delta^{'p} \in (0, \delta^{'p*}]$. Since DMU_o is efficient in stage 2 of period p, we have: $\mu^{(p)*} y_o^p - \pi^{(p)*} z_o^p + u^{2p*} = 0$. In this case,

$P^{p}_{2_{\delta^{'p}}} = \mu^{(p)*}(1-\delta^{'p}) y_o^p - \pi^{(p)*}(1-\delta^{'p}) z_o^p + u^{1p*} = (\mu^{(p)*} y_o^p - \pi^{(p)*} z_o^p)(1-\delta^{'p}) +$
$u^{2p*} + \delta^{'p} u^{2p*} - \delta^{'p} u^{2p*} = (\mu^{(p)*} y_o^p - \pi^{(p)*} z_o^p + u^{2p*})(1-\delta^{'p}) - \delta^{'p} u^{2p*} = 0 + \delta^{'p} u^{2p*} = \delta^{'p} u^{2p*}$

Thus, if $\mathrm{f}\, u^{2p*} < 0$, then $P^{p}_{2_{\delta^{'p}}} < 0$. Therefore, $Z^{p}_{\delta^{'p}} \in \mathrm{int}\, T_{2p}$ and according to Definition 1, DMU_o in stage 2 of period p has DRS.

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[DOI: 10.71885/ijorlu-2023-1-646]

5-iii. If $u^{2p^*}=0$, DMU_o in stage 2 has CRS.

Proof. Suppose $(\mu^{(p)^*},\omega^{(p)^*},\pi^{(p)^*},u^{1p^*},u^{2p^*})$ is the optimal solution of model (3) and there are $\delta^{p^*}>0$, $\delta^{'p^*}>0$, such that

$$\begin{cases} Z^{p}{}_{\delta^{p}}=((1+\delta^{p})z_o^{p},(1+\delta^{p})y_o^{p}),\ \delta^{p}\in(0,\delta^{p^*}] \\ Z^{p}{}_{\delta^{'p}}=((1-\delta^{'p})z_o^{p},(1-\delta^{'p})y_o^{p}),\delta^{'p}\in(0,\delta^{'p^*}] \end{cases}$$

Since DMU_o is efficient in stage 2 of period p, we have $\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p}+u^{2p^*}=0$. In this case,

$$\begin{cases} P^{p}{}_{2\delta^{p}}=\mu^{(p)^*}(1+\delta^{p})y_o^{p}-\pi^{(p)^*}(1+\delta^{p})z_o^{p}+u^{1p^*}=(\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p})(1+\delta^{p})+ \\ u^{2p^*}+\delta^{p}u^{2p^*}-\delta^{p}u^{2p^*}=(\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p}+u^{2p^*})(1+\delta^{p})-\delta^{p}u^{2p^*}=0 \\ \\ P^{p}{}_{2\delta^{'p}}=\mu^{(p)^*}(1-\delta^{'p})y_o^{p}-\pi^{(p)^*}(1-\delta^{'p})z_o^{p}+u^{1p^*}=(\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p})(1-\delta^{'p})+ \\ u^{2p^*}+\delta^{'p}u^{2p^*}-\delta^{'p}u^{2p^*}=(\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p}+u^{2p^*})(1-\delta^{'p})+\delta^{'p}u^{2p^*}=0 \end{cases}$$

Thus, if $u^{2p^*}=0$, then, $Z^{p}{}_{\delta^{p}}$ and $Z^{p}{}_{\delta^{'p}}$ are efficient in stage 1 of period p and according to Definition 1, DMU_o in stage 2 of period p has CRS.

So too, for determining the type of RTS in stage 2 of DMU_o in whole time periods, theorem 6 is introduced by using the sign $\sum_{p=1}^{q}u^{2p^*}$.

Theorem6. Suppose DMU_o is efficient in stage 2 of q time periods, i.e $E_o^{\mathrm{II}}=1$,

6-i. If $\sum_{p=1}^{q}u^{2p^*}>0$, DMU_o in stage 2 of q time periods has IRS.

Proof. Suppose $(\mu^{(p)^*},\omega^{(p)^*},\pi^{(p)^*},u^{1p^*},u^{2p^*})$ is the optimal solution of model (3) and there are $\delta^{p^*}>0,\ p=1,...,q$ such that, $Z^{p}{}_{\delta^{p}}=((1+\delta^{p})z_o^{p},(1+\delta^{p})y_o^{p}),\ \delta^{p}\in(0,\delta^{p^*}]$. Since DMU_o is efficient in stage 2 of q time periods, we have $\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p}+u^{2p^*}=0,\ p=1,...,q$. In this case,

$$P^{p}{}_{2\delta^{p}}=\mu^{(p)^*}(1+\delta^{p})y_o^{p}-\pi^{(p)^*}(1+\delta^{p})z_o^{p}+u^{1p^*}=(\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p})(1+\delta^{p})+$$

$$u^{2p^*}+\delta^{p}u^{2p^*}-\delta^{p}u^{2p^*}=(\mu^{(p)^*}y_o^{p}-\pi^{(p)^*}z_o^{p}+u^{2p^*})(1+\delta^{p})-\delta^{p}u^{2p^*}=$$

$$0-\delta^{p}u^{2p^*}=-\delta^{p}u^{2p^*},\ p=1,...,q\ \Rightarrow\sum_{p=1}^{q}P^{p}{}_{2\delta^{p}}=\sum_{p=1}^{q}-\delta^{p}u^{2p^*}$$

Thus, if $\sum_{p=1}^{q}u^{2p^*}>0$, then $\sum_{p=1}^{q}P^{p}{}_{2\delta^{p}}<0$. Therefore, $Z^{p}{}_{\delta^{p}}\in\operatorname{int}T_2$. This means that according to Definition 1, DMU_o in stage 2 over q time periods has IRS.

6-ii. If $\sum_{p=1}^{q}u^{2p^*}<0$, DMU_o in stage 2 of q time periods has DRS.

Proof. It is similar to the proof of **(6-i).**

6-iii. If $\sum_{p=1}^{q}u^{2p^*}=0$, DMU_o in stage 2 of q time periods has CRS.

Proof. It is similar to the proof of **(6-i)**.

In order to recognize the type of RTS in whole system of period p, the following theorem is introduced:

Theorem7. Suppose DMU_o is efficient in period p, i.e. $E_o^{\mathrm{s}(p)}=E_o^{\mathrm{I}(p)}=E_o^{\mathrm{II}(p)}=1$,

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[DOI: 10.71885/ijorlu-2023-1-646]

7-i. If $(u^{1p^*}+u^{2p^*})>0$, DMU_o has *IRS*.

Proof. Suppose $(\mu^{(p)^*},\omega^{(p)^*},\pi^{(p)^*},u^{1p^*},u^{2p^*})$ is the optimal solution of model (3) and there is $\delta^{p^*}>0$ such that $Z^p{}_{\delta^p}=((1+\delta^p)x_o^p,(1+\delta^p)z_o^p,(1+\delta^p)y_o^p),\ \delta^p\in(0,\delta^{p^*}]$. Since DMU_o is efficient in period p, we have: $\begin{cases}\pi^{(p)^*}z_o^p-\omega^{(p)^*}x_o^p+u^{1p^*}=0\\ \mu^{(p)^*}y_o^p-\pi^{(p)^*}z_o^p+u^{2p^*}=0\end{cases}$. In this case,

$$\begin{cases}P^p{}_{1_{\delta^p}}=\pi^{(p)^*}(1+\delta^p)z_o^p-\omega^{(p)^*}(1+\delta^p)x_o^p+u^{1p^*}=(\pi^{(p)^*}z_o^p-\omega^{(p)^*}x_o^p)(1+\delta^p)+\\ u^{1p^*}+\delta^p u^{1p^*}-\delta^p u^{1p^*}=(\pi^{(p)^*}z_o^p-\omega^{(p)^*}x_o^p+u^{1p^*})(1+\delta^p)-\delta^p u^{1p^*}=0-\delta^p u^{1p^*}=-\delta^p u^{1p^*}\\ \\ P^p{}_{2_{\delta^p}}=\mu^{(p)^*}(1+\delta^p)y_o^p-\pi^{(p)^*}(1+\delta^p)z_o^p+u^{1p^*}=(\mu^{(p)^*}y_o^p-\pi^{(p)^*}z_o^p)(1+\delta^p)+\\ u^{2p^*}+\delta^p u^{2p^*}-\delta^p u^{2p^*}=(\mu^{(p)^*}y_o^p-\pi^{(p)^*}z_o^p+u^{2p^*})(1+\delta^p)-\delta^p u^{2p^*}=0-\delta^p u^{2p^*}=-\delta^p u^{2p^*}\\ \Rightarrow P^p{}_{1_{\delta^p}}+P^p{}_{2_{\delta^p}}=-\delta^p(u^{1p^*}+u^{2p^*})\end{cases}$$

Thus, if $u^{1p^*}+u^{2p^*}>0$, then $P^p{}_{1_{\delta^p}}+P^p{}_{2_{\delta^p}}<0$. Hence $Z^p{}_{\delta^p}\in\operatorname{int}T$ and according to Definition 1, DMU_o in period p has *IRS*.

7-ii. If $(u^{1p^*}+u^{2p^*})<0$, DMU_o has *DRS*.

Proof. Suppose $(\mu^{(p)^*},\omega^{(p)^*},\pi^{(p)^*},u^{1p^*},u^{2p^*})$ is the optimal solution of model (3), and for period p, there is $\delta'^{p^*}>0$ and $Z^p{}_{\delta'^p}=((1-\delta'^p)x_o^p,(1-\delta'^p)z_o^p,(1-\delta'^p)y_o^p),\ \delta'^p\in(0,\delta'^{p^*}]$. Since DMU_o is efficient in period p, we have:

$$\begin{cases}\pi^{(p)^*}z_o^p-\omega^{(p)^*}x_o^p+u^{1p^*}=0\\ \mu^{(p)^*}y_o^p-\pi^{(p)^*}z_o^p+u^{2p^*}=0\end{cases}$$

In this case,

$$\begin{cases}P^p{}_{1_{\delta'^p}}=\pi^{(p)^*}(1-\delta'^p)z_o^p-\omega^{(p)^*}(1-\delta'^p)x_o^p+u^{1p^*}=(\pi^{(p)^*}z_o^p-\omega^{(p)^*}x_o^p)(1-\delta'^p)+\\ u^{1p^*}+\delta'^p u^{1p^*}-\delta'^p u^{1p^*}=(\pi^{(p)^*}z_o^p-\omega^{(p)^*}x_o^p+u^{1p^*})(1-\delta'^p)+\delta'^p u^{1p^*}=0+\delta'^p u^{1p^*}=\delta'^p u^{1p^*}\\ \\ P^p{}_{2_{\delta'^p}}=\mu^{(p)^*}(1-\delta'^p)y_o^p-\pi^{(p)^*}(1-\delta'^p)z_o^p+u^{1p^*}=(\mu^{(p)^*}y_o^p-\pi^{(p)^*}z_o^p)(1-\delta'^p)+\\ u^{2p^*}+\delta'^p u^{2p^*}-\delta'^p u^{2p^*}=(\mu^{(p)^*}y_o^p-\pi^{(p)^*}z_o^p+u^{2p^*})(1-\delta'^p)-\delta'^p u^{2p^*}=0+\delta'^p u^{2p^*}=\delta'^p u^{2p^*}\\ \\ \Rightarrow P^p{}_{1_{\delta'^p}}+P^p{}_{2_{\delta'^p}}=\delta'^p(u^{1p^*}+u^{2p^*})\end{cases}$$

Thus, if $(u^{1p^*}+u^{2p^*})<0$, then $P^p{}_{1_{\delta'^p}}+P^p{}_{2_{\delta'^p}}<0$. Therefore, $Z^p{}_{\delta'^p}\in\operatorname{int}T_p$ and according to Definition 1, DMU_o in period p has *DRS*.

7-iii. If $u^{1p^*}=u^{2p^*}=0$, DMU_o has *CRS*.

Proof. Suppose $(\mu^{(p)^*},\omega^{(p)^*},\pi^{(p)^*},u^{1p^*},u^{2p^*})$ is the optimal solution of model (3) and there are $\delta^{p^*}>0$, $\delta'^{p^*}>0$, such that

$$\begin{cases}Z^p{}_{\delta^p}=((1+\delta^p)x_o^p,(1+\delta^p)z_o^p,(1+\delta^p)y_o^p),\ \delta^p\in(0,\delta^{p^*}]\\ Z^p{}_{\delta'^p}=((1-\delta'^p)x_o^p,(1-\delta'^p)z_o^p,(1-\delta'^p)y_o^p),\delta'^p\in(0,\delta'^{p^*}]\end{cases}$$

Since DMU_o is efficient in period p, we have: $\begin{cases}\pi^{(p)^*}z_o^p-\omega^{(p)^*}x_o^p+u^{1p^*}=0\\ \mu^{(p)^*}y_o^p-\pi^{(p)^*}z_o^p+u^{2p^*}=0\\ \mu^{(p)^*}y_o^p-\omega^{(p)^*}x_o^p+u^{1p^*}+u^{2p^*}=0\end{cases}$

In this case,

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$$\begin{cases} P^{p}{}_{1\delta^{p}} = \pi^{(p)*}(1+\delta^{p})z_o^{p} - \omega^{(p)*}(1+\delta^{p})\mathrm{x}_o^{p} + u^{1p*} = (\pi^{(p)*}z_o^{p} - \omega^{(p)*}\mathrm{x}_o^{p})(1+\delta^{p}) + u^{1p*} + \delta^{p}u^{1p*} - \delta^{p}u^{1p*} = \\ (\pi^{(p)*}z_o^{p} - \omega^{(p)*}\mathrm{x}_o^{p} + u^{1p*})(1+\delta^{p}) - \delta^{p}u^{1p*} = 0 \\ \\ P^{p}{}_{2\delta^{p}} = \mu^{(p)*}(1+\delta^{p})y_o^{p} - \pi^{(p)*}(1+\delta^{p})z_o^{p} + u^{1p*} = (\mu^{(p)*}y_o^{p} - \pi^{(p)*}z_o^{p})(1+\delta^{p}) + \\ u^{2p*} + \delta^{p}u^{2p*} - \delta^{p}u^{2p*} = (\mu^{(p)*}y_o^{p} - \pi^{(p)*}z_o^{p} + u^{2p*})(1+\delta^{p}) - \delta^{p}u^{2p*} = 0 \\ \Rightarrow P^{p}{}_{1\delta^{p}} + P^{p}{}_{2\delta^{p}} = 0 \\ \\ P^{p}{}_{1\delta'^{p}} = \pi^{(p)*}(1-\delta'^{p})z_o^{p} - \omega^{(p)*}(1-\delta'^{p})\mathrm{x}_o^{p} + u^{1p*} = (\pi^{(p)*}z_o^{p} - \omega^{(p)*}\mathrm{x}_o^{p})(1-\delta'^{p}) + \\ u^{1p*} + \delta'^{p}u^{1p*} - \delta'^{p}u^{1p*} = (\pi^{(p)*}z_o^{p} - \omega^{(p)*}\mathrm{x}_o^{p} + u^{1p*})(1-\delta'^{p}) + \delta'^{p}u^{1p*} = 0 \\ \\ P^{p}{}_{2\delta'^{p}} = \mu^{(p)*}(1-\delta'^{p})y_o^{p} - \pi^{(p)*}(1-\delta'^{p})z_o^{p} + u^{1p*} = (\mu^{(p)*}y_o^{p} - \pi^{(p)*}z_o^{p})(1-\delta'^{p}) + \\ u^{2p*} + \delta'^{p}u^{2p*} - \delta'^{p}u^{2p*} = (\mu^{(p)*}y_o^{p} - \pi^{(p)*}z_o^{p} + u^{2p*})(1-\delta'^{p}) + \delta'^{p}u^{2p*} = 0 \\ \Rightarrow P^{p}{}_{1\delta'^{p}} + P^{p}{}_{2\delta'^{p}} = 0 \end{cases}$$

Therefore, if $u^{1p*} = u^{2p*} = 0$ then, $Z^{p}{}_{\delta^{p}}, Z^{p}{}_{\delta'^{p}}$ are efficient in period p and according to Definition 1, DMU_o in period p has *CRS*.

Theorem 8 use the sign $(\sum_{p=1}^{q} u^{1p*} + \sum_{p=1}^{q} u^{2p*})$ and Definition 1 to determine the type of RTS of DMU_o in whole time periods:

Theorem8. Suppose DMU_o is efficient over q time periods, i.e $E_o^s = E_o^{\mathrm{I}} = E_o^{\mathrm{II}} = 1$,

8-i. If $(\sum_{p=1}^{q} u^{1p*} + \sum_{p=1}^{q} u^{2p*}) > 0$, DMU_o over q time periods has *IRS*.

Proof. Suppose $(\mu^{(p)*}, \omega^{(p)*}, \pi^{(p)*}, u^{1p*}, u^{2p*})$ is optimal solution of model (3) and there are $\delta^{p*} > 0,\ p = 1,...,q$ such that $Z^{p}{}_{\delta^{p}} = ((1+\delta^{p})x_o^{p}, (1+\delta^{p})z_o^{p}, (1+\delta^{p})y_o^{p}),\ \delta^{p} \in (0, \delta^{p*}]$. Since DMU_o is efficient over q time periods, we have:

$$\begin{cases} \pi^{(p)*}z_o^{p} - \omega^{(p)*}x_o^{p} + u^{1p*} = 0,\ p = 1,...,q \\ \mu^{(p)*}y_o^{p} - \pi^{(p)*}z_o^{p} + u^{2p*} = 0,\ p = 1,...,q \\ \mu^{(p)*}y_o^{p} - \omega^{(p)*}x_o^{p} + u^{1p*} + u^{2p*} = 0,\ p = 1,...,q \end{cases}$$

In this case,

$$\begin{cases} P^{p}{}_{1\delta^{p}} = \pi^{(p)*}(1+\delta^{p})z_o^{p} - \omega^{(p)*}(1+\delta^{p})x_o^{p} + u^{1p*} = (\pi^{(p)*}z_o^{p} - \omega^{(p)*}x_o^{p})(1+\delta^{p}) + \\ u^{1p*} + \delta^{p}u^{1p*} - \delta^{p}u^{1p*} = (\pi^{(p)*}z_o^{p} - \omega^{(p)*}\mathrm{x}_o + u^{1p*})(1+\delta^{p})x_o^{p} - \delta^{p}u^{1p*} = 0 - \delta^{p}u^{1p*} = -\delta^{p}u^{1p*} \\ \\ P^{p}{}_{2\delta^{p}} = \mu^{(p)*}(1+\delta^{p})y_o^{p} - \pi^{(p)*}(1+\delta^{p})z_o^{p} + u^{2p*} = (\mu^{(p)*}y_o^{p} - \pi^{(p)*}z_o^{p})(1+\delta^{p}) + \\ u^{2p*} + \delta^{p}u^{2p*} - \delta^{p}u^{2p*} = (\mu^{(p)*}y_o^{p} - \pi^{(p)*}z_o^{p} + u^{2p*})(1+\delta^{p}) - \delta^{p}u^{2p*} = 0 - \delta^{p}u^{2p*} = -\delta^{p}u^{2p*} \\ \\ P^{p}{}_{1\delta^{p}} + P^{p}{}_{2\delta^{p}} = -\delta^{p}u^{1p*} - \delta^{p}u^{2p*} = -\delta^{p}(u^{1p*} + u^{2p*}),\ p = 1,...,q \\ \\ \Rightarrow \sum_{p=1}^{q}(P^{p}{}_{1\delta^{p}} + P^{p}{}_{2\delta^{p}}) = \sum_{p=1}^{q}(-\delta^{p}(u^{1p*} + u^{2p*})) = -\sum_{p=1}^{q}\delta^{p}(u^{1p*} + u^{2p*}) \end{cases}$$

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Therefore, if $(\sum_{p=1}^{q} u^{1p^*} + \sum_{p=1}^{q} u^{2p^*}) > 0$, then $\sum_{p=1}^{q} (P^{p}_{1_{\delta^p}} + P^{p}_{2_{\delta^p}}) < 0$ and thus $Z^{p}_{\delta^p} \in \text{int}\, T, \delta^p \in (0, \delta^{p^*}]$. This means that according to Definition 1, DMU_o has IRS.

8-ii. If $(\sum_{p=1}^{q} u^{1p^*} + \sum_{p=1}^{q} u^{2p^*}) < 0$, DMU_o over q time periods has DRS.

Proof. It is similar to the proof of **(8-i).**

8-iii. If $\sum_{p=1}^{q} u^{1p^*} = \sum_{p=1}^{q} u^{2p^*} = 0$, DMU_o over q time periods has CRS.

Proof. It is similar to the proof of **(8-i).**

So, by using these theorems, it can be said that a multi-period two-stage system has IRS (or DRS or CRS) if and only if stage 1 and stage 2 have IRS (or DRS or CRS) in whole time periods. And if one stage has IRS and the other has DRS in each period p (or whole time periods), theorem 7 (or theorem 8) can identify RTS status of whole system in each period p (or whole time periods).

4 Case study

In this section, we will illustrate the proposed models with Taiwanese non-life insurance companies studied in Kao and Hwang [31] to compare the obtained results with their results. They surveyed the efficiency of 21 non-life insurance companies with the data in 2000, 2001, 2002 years. Each insurance companies can be viewed as *DMU* with two-stage structure. In this evaluation, inputs are "Operating expenses" and "Insurance expenses", intermediate products are "Direct written premiums" and "Reinsurance premiums". Also, "Underwriting profit" and "Investment profit" are considered as outputs.

4.1 Results of model (3)

The efficiencies of $DMUs$ under the VRS assumptions based on the model (3), are reported in the following Table:

Table 1. Results of Model (3)

DMU	period	Overall efficiency	Efficiency of stage 1	Efficiency of stage 2
1	1	0.5287	0.9930	0.5325
	2	0.5841	0.9732	0.6002
	3	1	1	1
	Whole period	0.9665	0.9987	0.9678
2	1	0.9295	0.9862	0.9425
	2	0.6317	1	0.6317
	3	0.6246	1	0.6246
	Whole period	0.8939	0.9879	0.9048
3	1	1	1	1
	2	1	1	1
	3	1	1	1

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	Whole period	1	1	1
4	1	0.6181	0.7931	0.7794
	2	0.6655	1	0.6655
	3	0.2946	0.8601	0.3425
	Whole period	0.6377	0.9761	0.6533
5	1	0.5641	0.9264	0.6089
	2	0.4873	0.7737	0.6289
	3	0.3953	0.7841	0.5041
	Whole period	0.5458	0.9050	0.6031
6	1	0.5656	0.8855	0.6388
	2	0.5952	0.8632	0.6895
	3	0.5068	0.7916	0.6403
	Whole period	0.5811	0.8568	0.6782
7	1	0.1365	0.9483	0.1439
	2	0.4312	1	0.4312
	3	0.2722	1	0.2722
	Whole period	0.4038	0.9969	0.4050
8	1	0.5800	0.8392	0.6912
	2	0.3922	0.8061	0.4866
	3	0.7789	0.7789	1
	Whole period	0.7496	0.7831	0.9572
9	1	0.3967	0.8796	0.4510
	2	0.3346	0.7791	0.4295
	3	0.3042	0.6914	0.4401
	Whole period	0.3882	0.8635	0.4496
10	1	0.8847	0.9778	0.9047
	2	0.7452	0.9851	0.7565
	3	0.7771	0.8836	0.8794
	Whole period	0.8709	0.9718	0.8962
11	1	0.8577	0.9583	0.8951
	2	0.9193	0.9193	1
	3	0.4115	0.8495	0.4844
	Whole period	0.8683	0.9154	0.9486
12	1	0.6008	0.8837	0.6799
	2	0.3863	0.7555	0.5114
	3	0.3743	0.6571	0.5679
	Whole period	0.5759	0.8630	0.6673
13	1	1	1	1
	2	1	1	1
	3	0.6762	1	0.6762
	Whole period	0.9800	1	0.9800
14	1	0.4572	0.8198	0.5577
	2	0.3510	0.8866	0.3959
	3	0.3710	0.9283	0.3996
	Whole period	0.4506	0.8260	0.5456
15	1	0.8432	0.9623	0.8762
	2	0.7441	0.7441	1
	3	0.6559	0.6559	1
	Whole period	0.8258	0.9311	0.8869
16	1	0.5167	0.8416	0.6140
	2	0.5304	0.8445	0.6280
	3	0.3028	0.7492	0.4042
	Whole period	0.5167	0.8416	0.6140
17	1	0.6509	1	0.6509
	2	0.3808	1	0.3808
	3	0.9196	1	0.9196
	Whole period	0.9151	1	0.9151

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18	1	0.6473	0.8061	0.8030
	2	0.8349	0.8349	1
	3	0.6721	0.6721	1
	Whole period	0.8334	0.8341	0.9992
19	1	1	1	1
	2	1	1	1
	3	1	1	1
	Whole period	1	1	1
20	1	0.8595	1	0.8595
	2	0.6838	1	0.6838
	3	0.0433	0.7773	0.0557
	Whole period	0.8574	0.9996	0.8578
21	1	0.8743	0.8743	1
	2	0.1173	0.6509	0.1802
	3	0.0994	0.6621	0.1501
	Whole period	0.8612	0.8706	0.9892

In Table 1, the columns 3, 4 and 5 indicate the overall efficiencies and efficiency of stages respectively. In period1, $DMUs$ $3,13,17,19,20$ in stage 1 and $DMUs$ $3,13,19,21$ in stage 2 are efficient. Hence, $DMUs$ $3,13,19$ are overall efficient. Between inefficient $DMUs$, in whole system and stage 2 the highest efficiency belongs to DMU_2 and the lowest efficiency belongs to DMU_7. Also, DMU_4, DMU_1 have the best and the lowest efficiencies in stage 1 of period 1. It is clear that $DMUs$ $3,13,19$ are overall efficient in period 2. Then, DMU_{10} has the best efficiency and DMU_{21} has the lowest efficiency in stage 1 and stage 2 of period 2. In period 3, DMUs 1, 3, 19 are overall efficient. Between inefficient $DMUs$, DMU_{20} has the worst efficiency in stage 2 and whole system. In period 3, the best efficiency in stage 2 and whole system belongs to DMU_{17}. In stage 1, DMU_{14}, DMU_{15} have the highest and the lowest efficiency, respectively. In whole time periods, $DMUs$ $3,19$ are efficient, but all of $DMUs$ are inefficient in Kao and Hwang model. Between inefficient $DMUs$, the highest and the lowest efficiencies belong to DMU_1, DMU_9 with scores 0.9665, 0.3882, respectively. In stage 1 $DMUs$ $3,13,17,19$ are efficient. Between inefficient $DMUs$, DMU_{20}, DMU_{17} have the best efficiency score in stage 1 and stage 2, respectively. Also, the lowest efficiency belongs to DMU_8 in stage 1 and DMU_7 in stage 2. Note that, efficient $DMUs$ in Kao and Hwang model are efficient under our model. According to the Table 3, the overall efficiencies obtained from Kao and Hwang Model [31] (i.e E_{oCRS}), is less than or equal the overall efficiencies obtained from Model (3), i.e $E_{oCRS} \le E_o^s$. Also, Table 2 shows the type of return to scale for the VRS efficient $DMUs$ in stage 1, stage 2 and system corresponding to each period and over 3 time periods:

Table 2. The type of RTS

DMU	Period	RTS(stage1)	RTS(stage2)	RTS (system)
1	1	-	-	-
	2	-	-	-
	3	DRS	DRS	DRS
	(Whole periods)	-	-	-
2	1	-	-	-
	2		-	-

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	3 (Whole periods)	*DRS* *DRS* *DRS* -	- -	- -
3	1 2 3 (Whole periods)	*DRS* *DRS* *DRS* *DRS*	*CRS* *CRS* *DRS* *DRS*	*DRS* *DRS* *DRS* *DRS*
4	1 2 3 (Whole periods)	- *IRS* - -	- - - -	- - - -
7	1 2 3 (Whole periods)	- *DRS* *IRS* -	- - - -	- - - -
8	1 2 3 (Whole periods)	- - - -	- - *DRS* -	- - - -
11	1 2 3 (Whole periods)	- - - -	- - *DRS* -	- - - -
13	1 2 3 (Whole periods)	*IRS* *DRS* *DRS* -	*DRS* *DRS* - -	*DRS* - - -
15	1 2 3 (Whole periods)	- - - -	- *DRS* *DRS* -	- - - -
17	1 2 3 (Whole periods)	*IRS* *IRS* *IRS* *IRS*	- - - -	- - - -
18	1 2 3 (Whole periods)	- - - -	- *IRS* *DRS* -	- - - -
19	1 2 3 (Whole periods)	*IRS* *IRS* *IRS* *IRS*	*IRS* *IRS* *DRS* *IRS*	*IRS* *IRS* *DRS* *IRS*
20	1 2 3 (Whole periods)	*IRS* *IRS* - -	- - - -	- - - -
21	1 2 3 (Whole periods)	- - - -	*IRS* - - -	- - - -

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In this Table, The columns 3, 4 and 5 indicate the type of RTS corresponding to stage 1, stage 2 and system in time periods 1, 2 and 3 and the whole time periods, respectively. As can be seen in the column 3, DMU_3 has DRS in stage 1 of each period, thus based on the Theorem 4, this DMU has DRS in stage 1 over whole time periods. Also, this DMU has CRS in stage 2 of period 1 and 2 and DRS in stage 2 of period 3. Thus DMU_3 has DRS in stage 2 over whole time periods. Since DMU has DRS in each period of whole system, hence, DMU_3 has DRS in whole system over whole time periods. DMU_{19}, has IRS in stage 1, 2 of period 1, 2 and IRS in stage 1 of period 3 and DRS in stage 2 of period 3. Thus, in whole system, this DMU has IRS in period 1 and 2 and DRS in period 3. Therefore, DMU_{19} has IRS in whole system over whole time periods. DMU_{17} has IRS in each period and whole time periods. In period 1, DMU_{13} has IRS in Stage 1 and DRS in Stage 2, thus based on the Theorem4, this DMU has DRS in the whole system. Also, DMU_{20} in stage 1 of period 1 and 2 has IRS and DMU_{21} has IRS in stage 2 of period 1. In period 2, $DMUs$ 2, 4, 7 have DRS, IRS and DRS in stage 1 and DMU_{15}, DMU_{18} have DRS and IRS, respectively. DMU_1 has DRS in stage 1, 2 of period 3, therefore, this DMU has DRS in the whole system. $DMUs$ 2, 7, 13 have DRS, IRS and DRS in stage 1 of period 3. And also, $DMUs$ 8, 11, 15, 18 have DRS in stage 2 of period 3. We can make the similar interpretations for other $DMUs$.

5 Conclusions

The traditional two-stage DEA models measure the efficiency of two-stage systems at the ascertained time, while the calculation of the efficiency of these systems is particularly crucial during the multi-period time. In this paper, we proposed a generalized model, to measure the general efficiency of a multi-period two-stage system under the VRS assumption so that the data substance is taken into account in varied time periods. We also decomposed the overall efficiency to the product of the efficiencies of stages in each period (and whole time periods). So, the efficiency in general (and that of the stages) in whole time periods was decomposed to the weighted average of the overall efficiency (and efficiencies of the stages) in each period. The proposed model determined the status of RTS in efficient DMUs in stage 1, 2 and the entire system in period p and over q time periods. The proposed model can identify sources of inefficiency by using the obtained efficiency decompositions. Eventually, we utilized a case of related to Taiwanese non-life insurance companies to illustrate the proposed model. For future study, one can generalize some models to measure the efficiency of a multi-period multi-stage system under the (VRS) and determine the type of returns to scale of each stage in period p and over the whole q time period. Moreover, further research on extending the proposed method to measure RTS of the same systems, in the presence of undesirable outputs is a motivating course for future research.

References

1. Charnes, A., Cooper, W.W. & Rhodes, E. (1978). Measuring the efficiency of decision making units, European Journal of Operational Research 2, 429-444.

[Downloaded from ijaor.ir on 2026-11-28]

[DOI: 10.71885/ijorlu-2023-1-646]

2. Seiford, L.M., & Zhu, J. (1999). Profitability and Marketability of the Top 55 US Commercial Banks, Management Science 45, 1270-1288.
3. Kao, C., &Hwang, S.N. (2008). Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan, European Journal of Operational Research 185, 418–429.
4. Chen, Y., Cook, W.D., Li, N. & Zhu, J. (2009). Additive efficiency decomposition in two-stage DEA, European Journal of Operational Research 196, 1170–1176.
5. Wang, Y.M., & Chin, K.S. (2010). Some alternative DEA models for two-stage process, Expert Systems with Applications, 37, 8799–8808.
6. Wu, J., Zhu, Q., Chu, J., & Liang L, L. (2015). Two-Stage Network Structures with Undesirable Intermediate Outputs Reused: A DEA Based Approach, Computational Economics 46(3), 455–477.
7. Fei, Y., Bi, G., Song, W., & Luo, Y. (2019). Measuring the Efficiency of Two-Stage Production Process in the Presence of Undesirable Outputs, Computational Economics 54, 1343-1358.
8. Chu, J., Wu, J., Zhu, Q., An, Q., & Xiong, B. (2016). Analysis of China's Regional Eco-efficiency: A DEA two- stage Network approach with equitable efficiency decomposition, Computational Economics 54, 1263-1285.
9. Despotis, D.K., Sotiros, D., & Koronakos, G.R. (2016). Composition versus decomposition in two-stage network DEA: a reverse approach, Journal of Productivity Analysis 45, 71–87.
10. Wu, J., Zhu, Q., Ji, X., Chu, J., & Liang, L. (2016). Two-stage network processes with shared resources and resources recovered from undesirable outputs, European Journal of Operational Research 251(1), 182–197, 2016.
11. Guo, C., & Zhu, J. (2017). Non-cooperative Two-stage Network DEA Model: Linear vs. Parametric Linear, European Journal of Operation Research, 258(1), 398-400.
12. Izadikhah, M., Tavana, M., Caprio, D.D., Francisco, J., & Santos-Arteaga, F.J. (2018). A novel two-stage DEA production model with freely distributed initial inputs and shared intermediate outputs, Expert Systems with Applications 99(1), 213-230.
13. Xueqin, C. ,Yongjun, L., Feng, C. , & Liang, L. (2021). Efficiency evaluation of commercial banks based on two-stage DEA model with undesirable output, Systems Engineering - Theory & Practice, 41(3), 636-648.
14. Akbarian, D. (2021). Network DEA based on DEA-ratio, Financ Innov 7(73) https://doi.org/10.1186/s40854-021-00278-6.
15. _Hosseini_Monfared, S.N, Hosseinzadeh_Lotfi , F., Mozaffari , M.R, & _Rostamy_Malkhalifeh, M. (2021). Classifying flexible and integer data in two-stage network data envelopment analysis, Journal of Applied Research on Industrial Engineering, 8(3), 270-291.
16. Amirteimoori, A. (2013). A DEA two-stage decision processes with shared resources, Central European Journal of Operations Research 21, 141–151. [14] Benicio, J., & De Mello, J.C. (2019). Different type of returns to scale in DEA, Pesquisa Operational 39(2), 245-260.
17. Aviles-Sacoto, S., Cook, W.D., Imanirad, R., & Zhu, J. (2015). Two-stage network DEA: when intermediate measures can be treated as outputs from the second stage, Journal of the Operational Research Society, 1–10.
18. Chu, J., & Zhu, J. (2021). Production scale-based two-stage network data envelopment analysis, European Journal of Operational Research, 294(1), 283-294.
19. Hong, J., Tamakloe, R., Tak, J., &Park, D. (2020). Two-Stage Double Bootstrap Data Envelopment Analysis for Efficiency Evaluation of Shared-Bicycle Stations in Urban Cities. 2674(6)211-224.
20. Liu, W., Zhou, Z., Ma, C., Liu, D., & Shen, W. (2015). Two-stage DEA Models with undesirable input intermediate outputs, Omega 56, 74-87.
21. Yu, Y., & Shi, Q. (2014). Two - stage DEA model with additional input in the second stage and part of intermediate products as final output, Expert Systems with Applications 41(15), 6570–6574.
22. Zha, Y., & Liang, L. (2010). Two-stage cooperation model with input freely distributed among the stages, European Journal of Operational Research 205,332–338.
23. Portela, M.C.S., Camanho, A.S., & Borges, D. (2012). Performance assessment of secondary schools: The snapshot of a country taken by DEA, Journal of the Operational Research Society 63, 1098–1115.
24. Kao, C., & Liu, S.T. (2014). Multi-period efficiency measurement in data envelopment analysis: The case of Taiwanese commercial banks, Omega 47, 90–98.
25. Razavi Hajiagha, H., Hashemi, S.H., Mahdiraji, H.A., & Azaddel, J. (2015). Multi-period data envelopment analysis based on Chebyshev inequality bounds, Expert Systems with Applications 42(21), 7759–7767.

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[DOI: 10.71885/ijorlu-2023-1-646]

26. Bansel, P., & Mehra, A. (2018). Multi-period additive efficiency measurement in data envelopment analysis with non-positive and undesirable data, Opesearch, 55, 642-661.
27. Jahani Sayyad Noveiri, M., & Kordrostami, S. (2018). Detecting the multi-period performance and efficiency changes of systems with undesirable outputs, Discrete Mathematics, Algorithms and Applications 10(3).
28. Tamatam, R., Dutta, P., Dutta, G., & Lessmann, S. (2019). Efficiency analysis of Indian banking industry over the period 2008–2017 using data envelopment analysis, Benchmarking: An International Journal, 26(8), 2417-2442.
29. Kordrostami, S., & Jahani Sayyed Noveiri, M. (2017). Evaluating the efficiency of firms with negative data in multi-period systems, International Journal of Industrial Mathematics 9(1), 27-35.
30. Kordrostami S., & Jahani Sayyed Noveiri, M. (2017). Evaluating the multi-period systems efficiency in the presence of fuzzy data, Fuzzy Information and Engineering 9(3), 281-298.
31. Kao, C., & Hwang, S.N. (2014). Multi-period efficiency and Malmquist productivity index in two-stage production systems, European Journal of Operational Research 232, 512–521.
32. Esfidani, S., Hosseinzadeh Lotfi, F., Razavyan, S., & Ebrahimnejad, A. (2018). Efficiency changes index in the network data envelopment analysis with non-radial model, Asian-European Journal of Mathematics 13(1).
33. Tohidnia, S., & Tohidi, G. (2019). Estimating multi-period global cost efficiency and productivity changes of systems with network structures, Journal of Industrial Engineering International 15, 171-179.
34. Zhou, X., Xu, Z., Chai, J., Yao, L., Wang, S., & Lev, B. (2019). Efficiency evaluation for banking systems under uncertainty: A multi-period three-stage DEA model, Omega 85, 68-82.
35. Esmaeilzadeh, A., & Kazemi Matin, R. (2019). Multi-period efficiency measurement of network production systems, Measurement, 134, 835-844.
36. Banker, R.D., & Thrall, R.M. (1992). Estimation of returns to scale using data envelopment Analysis, European Journal of Operational Research 62(1), 74–84.
37. Färe, R., & Grosskopf, S. (1994). Estimation of returns to scale using data envelopment analysis: A Comment, Journal of Operational Research 79, 379-382.
38. Khodabakhshi, M., Gholami, Y., & Kheirollahi, H. (2010). A additive model approach for estimating returns to scale in imprecise data envelopment analysis, Applied Mathematical Modelling 34, 1247– 1257.
39. Banker, R.D., Charnes, A., & Cooper, W.W. (1984). Some models for estimating technical and scale efficiencies in data envelopment analysis, Management Science **30**, 1078–1092.
40. Fanati Rashidi, S. (2019). Determination of Right/Left return to Scale in Two-Stage Processes Based on Dual Simplex. Int. J. Data Envelopment Analysis (ISSN 2345-458X) 7(3), 2019.
41. Gerami, J. (2020). Return to scale in two-stage networks based on two different approaches. 10.22105/DMOR.2020.103204.
42. Jahanshahloo, G.R. & Khodabakhshi, M. (2003). Using input-output orientation model for determining most productive scale size in DEA, Applied Mathematics and Computation 146, 849-855.
43. Seiford, L.M., & Zhu, J. (1999). An investigation of returns to scale in Data Envelopment Analysis, Omega 27, 1-11.
44. Soleimani-damaneh, M., Jahanshahloo, G.R., Mehrabian, S., & Hasannasab, M. (2010). Returns to scale and scale elasticity in the presence of weight restrictions and alternative solutions, Knowledge Based System 23, 86-93.
45. Soleimani-damaneh, M. (2012). On a basic definition of returns to scale, Operations Research Letters 40(24), 144-147.
46. Wei, Q.L., & Yan, H. (2004). Congestion and returns to scale in data envelopment analysis, European Journal of Operational Research 153, 641-660.
47. Krivonozhko, V.E., orsund, F.R., & Lychev, A.V. (2012). Returns-to-scale properties in DEA models: The fundamental role of interior points, Journal of Productivity Analysis 38, 121-130.
48. Benicio, J., & De Mello, J.C. (2019). Different type of returns to scale in DEA, Pesquisa Operational 39(2), 245-260.
49. Khaleghi, M., Jahanshahloo, G.R., Zohrebandian, M., & Hosseinzadeh Lotfi, F. (2012). Returns to scale and scale elasticity in two-stage DEA, Mathematical and Computational Applications, 17(3), 193-202.
50. Zhang, Q., &Yang, Z. (2015). Returns to scale of two-stage production process, Computers and Industrial Engineering 90, 259-268.
51. Fakhr Mousavi, S.M, Amirteimoori, A., Kordrostami, S., & Vaez-Ghasemi , M. (2021). Non-radial two-stage network DEA model to estimate returns to scale, Journal of Modelling in Management, ISSN: 1746-5664.

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[DOI: 10.71885/ijorlu-2023-1-646]

52. Yang, J., & Fang, L. (2022). Average lexicographic efficiency decomposition in two-stage data envelopment analysis: an application to China's regional high-tech innovation systems. Ann Oper Res 312, 1051–1093 (2022). https://doi.org/10.1007/s10479-021-04427-z.
53. Hosseinzadeh Lotfi F., Sanei M., Hosseinzadeh A., Niroomand S., & Mahmoodirad A. (2023). Uncertainty in Data Envelopment Analysis. Fuzzy and Belief Degree-Based Uncertainties, 1st Edition, 9780323994453.
54. Yu X. (2023). An assessment of the green development efficiency of industrial parks in China: Based on non-desired output and non-radial DEA model. Structural Change and Economic Dynamics, 66, 81-88.
55. Patyal V.S., Kumar R., Lamba K., & Sunil Maheshwari S. (2023). Performance evaluation of Indian electricity distribution companies: An integrated DEA-IRP-TOPSIS approach. Energy Economics, 106796.
56. Wan L., Zhang L., Chiu Y.H., Pang Q., Qian Luo Q., & Zhen Shi Z. (2023). A bootstrapping dynamic two-stage SBM model: An application to assess industrial water use and health risk systems. Science of the Total Environment, 164813.
57. Zhang X., Xia Q., & Wei F. (2023). Efficiency evaluation of two-stage parallel-series structures with fixed-sum outputs: An approach based on SMAA and DEA. Expert Systems with Applications, 227, 120264.

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